Section 3.3: A CALCULUS REVIEW.

Throughout this chapter, there are some ideas from the multi-dimensional calculus which we will use. In addition, the following notational agreements should be recalled :

$$\text{grad } u = \nabla u = \{\partial u/\partial x,\ \partial u/\partial y\},$$

$$\text{div } v = \nabla \cdot v = \partial v_1/\partial x + \partial v_2/\partial y, \qquad \text{(often called } \textit{del } v\text{)}$$

$$\nabla^2 u = \nabla \cdot \nabla u = \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 \quad \text{in rectangular coordinates,}$$

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \Theta^2} \qquad \text{in polar coordinates,}$$

$$\partial u/\partial \eta = \langle \nabla u, \eta \rangle \qquad \text{where } \eta \text{ is a vector, and}$$

$$\text{curl } F = \det\begin{pmatrix} i & j & k \\ \partial(\ )/\partial x & \partial(\ )/\partial y & \partial(\ )/\partial z \\ F_1 & F_2 & F_3 \end{pmatrix}$$

$$= \{\partial F_3/\partial y - \partial F_2/\partial z,\ \partial F_1/\partial z - \partial F_3/\partial x,\ \partial F_2/\partial x - \partial F_1/\partial y\}.$$

There is also information from the integral calculus which we should recall. Some of the important ideas involve methods of calculating surface integrals. Recall that if D is an open, connected region in the plane and f is a function defined on D which has continuous partial derivatives, and if S is the surface which is the graph of f, then we can define a surface integral

$$\iint_S H(x,y,z)\, dA$$

where H is a continuous function with domain S. This integral over the surface S in R^3 can be evaluated by changing it to an integral over the 2-dimensional region D as

$$\iint_D H(x,y,f(x,y)) \sqrt{[(df/dx)^2+(df/dy)^2+1]}\ dx\, dy.$$

For such a surface, a unit normal is given by

$$\eta = \{-\partial f/\partial x, -\partial f/\partial y, 1\} / \sqrt{[(df/dx)^2+(df/dy)^2+1]}\,.$$

The unit normal to a plane curve described by {x(t), y(t)} is

$$\eta = \{y'(t), -x'(t)\} / \sqrt{[\,x'(t)^2 + y'(t)^2\,]}.$$

The following are fundamental ideas used in vector analysis.

STOKES THEOREM IN 2D. Suppose that D is a region in the plane with a piece-wise smooth boundary and that F(x,y) ={P(x,y),Q(x,y)} has continuous second partial derivatives and is a function from R^2 to R^2. Then

$$\int_{\partial D} P dx + Q dy \quad = \quad \iint_D (\partial Q/\partial x \; - \; \partial P/\partial y) \, dx \, dy.$$

STOKES THEOREM IN 3D. Suppose that D is a smooth surface with unit normal η and has a smooth boundary. Suppose, also, that F is a function from R^3 to R^3 which has continuous second partial derivatives. Then

$$\int_{\partial D} < F, dR > \quad = \quad \iint_D < \text{curl } F, \eta > dA.$$

DIVERGENCE THEOREM. With proper suppositions on D and F

$$\int_{\partial D} < F, \eta > ds \quad = \quad \iint_D \text{div } F \, dA = \iint_D \qquad F \, dA.$$

in two dimensions, while

$$\iint_{\partial D} < F, \eta > dA \quad = \quad \iiint_D \text{div } F \, dV = \iiint_D \qquad F \, dV.$$

in three dimensions.

REMARKS.

1. Here is a suggestion for the proof of the Divergence Theorem which uses Stokes Theorem: Take R(t) to be a parameterization of the boundary given by$\{ \Phi(t), \Psi(t)\}$. Then $dR(t) = \{ \Phi'(t), \Psi'(t) \} \, dt$ and $\eta \, ds = \{\Psi'(t), -\Phi'(t)\} \, dt$. Thus,

$$\iint_D \text{div } F \, dA \quad = \quad \iint_D \; [\partial F_1/\partial x + \partial F_2/\partial y] \, dA$$

$$= \iint_D \left[\frac{\partial F_1}{\partial x} - \frac{\partial(-F_2)}{\partial y}\right] dx \, dy.$$

Recall that Stokes theorem in 2-D says that this last two dimensional integral can be changed to a line integral:

$$\iint_D \left[\frac{\partial F_1}{\partial x} - \frac{\partial -F_2}{\partial y}\right] dx \, dy$$

$$= \int_{\partial D} [-F_2 \, dx + F_1 \, dy]$$

$$= \int_{\partial D} < \{F_1 , F_2\}, \{\Psi', -\Phi'\} > dt$$

$$= \int_{\partial D} < F , \eta > ds . \qquad\qquad f$$

2. The Divergence Theorem is a generalization of the fundamental theorem of integral calculus in the following sense: Let D be the rectangle (a,b)x(c,d) and F(x,y) ={u(x),0}. Then div F =u'(x) and

$$(d-c) \int_a^b u'(x)\, dx = \iint_D \text{div } F \, dA.$$

By the Divergence Theorem, this latter is

$$\int_{\partial D} < F , \eta > ds = \int_a^b < \{u, 0\}, \{0,-1\} > dx + \int_c^d < \{u(b),0\} , \{1,0\} > dy$$

$$+ \int_a^b < \{u,0\} , \{0,1\} > |dx| + \int_c^d < \{u(a) , 0\} , \{-1,0\} > |dy|$$

$$= u(b)\,(d-c) - u(a)\,(d-c) = [u(b) - u(a)]\,(d-c).$$

Thus, in one dimension, the divergence theorem specializes to

$$\int_a^b u'(x)\, dx = u(b) - u(a).$$

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EXERCISE:

1. Suppose that A = $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$**, B = {5,7}, and C = 11. Show that**

$$A \quad u + B \quad u + Cu = \frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 3\frac{\partial^2 u}{\partial y^2} + 5\frac{\partial u}{\partial x} + 7\frac{\partial u}{\partial y} + 11\, u.$$

2.Suppose that $L[u] = \frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + 3\frac{\partial^2 u}{\partial y^2} + 4\frac{\partial u}{\partial x} + 5\frac{\partial u}{\partial y} + 6\, u$**. Find A, B, and C such that L[u] = A u + B u + Cu.**

3. Suppose the P(x,y) and Q(x,y) have continuous partial derivatives and that F is defined by F(x,y,z) = {P(x,y), Q(x,y), 0}. Show that

$$< \text{curl } F, \{0,0,1)> = (\partial Q/\partial x - \partial P/\partial y).$$

4. Application of Stokes Theorem in the plane: Integrate < curl F, η > over D.

a. F(x,y) = {x,y}, D = $D_1(0)$ **(= the unit disk).**

b. F(x,y) = {-y,x}, D = $D_1(0)$**,**

c. F(x,y) = [3y, 5x}, D = $D_1(0)$**,** **ans:** 2π

d. F(x,y) = $\{0,x^2\}$**, D is the rectangle with vertices at {0,0}, {a,0}, {a,b}, and {0,b}.** **ans:** a^2b

e. $F(x,y) = \{3xy + y^2, 2xy + 5x^2\}$, $D = D_1(\{1,2\})$ (= the disk with radius 1 and center {1,2}). ans: 7π.

5. Application of Stokes Theorem in 3D: Integrate < curl F, η > over D.

a. $F(x,y,z) = \{x,y,z\}$, D is the upper half of the unit sphere. ans: 0.

b. $F(x,y,z) = \{z^2, 2x, -y^3\}$, D is as above. ans: 2π.

c. $F(x,y,z) = \{2z, -y, x\}$, D is the triangle with vertices at {2,0,0}, {0,2,0}, {0,0,2}. ans: 2.

d. $F(x,y,z) = \{x^4, xy, z^4\}$, D is the as above. ans: 4/3.

6. Application of the Divergence Theorem. Verify the Divergence Theorem on these regions.

a. $F(x,y,z) = \{x, y, z\}$, D = the unit sphere. ans: 4π.

b. $F(x,y,z) = \{x^2, y^2, z^2\}$, D = the unit sphere. ans: 0.

c. $F(x,y,z)= \{x,y,z\}$, D = the unit cube in the 1st octant. ans: 3.

d. $F(x,y,z) = \{x^2, -xz, z^2\}$, D = the unit cube in the 1st octant ans: 2.