Section 3.4: GREEN'S IDENTITIES

As students study the integration identities in the multi-dimensional calculus, the chief applications they see for these formulas are likely in the computation of work along a path, or flux through a solid, or rotation of a surface. In this section, we will show that these identities may be used to derive the formulas which are used to study other physical phenomenia. Also, one of the Green's identities is a multi-dimensional version of integration-by-parts. Recalling that integration-by-parts played such an important role in defining the adjoint of differential operators, it is no surprise that the corresponding identity plays a similar role here.

THEOREM (<u>GREEN'S FIRST IDENTITY</u>) Suppose that D is a region in the plane with a piecewise smooth boundary and that U and V have continuous second partial derivatives. Then,

$$\iint_D V \quad {}^2U \, dA + \iint_D < \quad V , \quad U > dA = \int_{\partial D} < V \quad U, \eta > ds.$$

Suggestion for proof: We will use the Divergence theorem. Let

$$P = V\, U_x \text{ and } Q = V\, U_y.$$

Then

$$\partial P/\partial x + \partial Q/\partial y = V\, U_{xx} + V_x\, U_x + V\, U_{yy} + V_y\, U_y$$

$$= V \quad {}^2U + < \quad V , \quad U >.$$

Is it clear how this is an application of the Divergence Theorem?

REMARK: Just as the divergence theorem generalizes the fundamental theorem of integral calculus, so Green's First Identity generalizes the integration-by-parts formulas: take V(x,y) = f(x) and U(x,y) = g(x) on the rectangle [a,b]x[c,d]. then

$$\iint_D V \quad {}^2U \, dA + \iint_D < \quad U , \quad V > dA$$

$$= \iint_D f(x)\, \partial^2 g/\partial x^2 + \iint_D < \{f', 0\} , \{g',0) > dA$$

$$= (d - c) \left[\int_a^b f(x)\, g''(x)\, dx + \int_a^b f'(x)\, g'(x)\, dx \right].$$

On the other hand,

$$\int_{\partial D} < V \quad U, \eta > ds = (d - c)\, [f(b)\, g'(b) - f(a)\, g'(a)].$$

THEOREM (<u>GREEN'S SECOND IDENTITY</u>) With D, U, and V as before

$$\iint_D [\quad {}^2U \quad V - U \quad {}^2V]\, dA = \int_{\partial D} [\partial U/\partial \eta \quad V - U \quad \partial V/\partial \eta]\, ds.$$

Suggestion of a proof. $\iint_D [\quad {}^2U \quad V - U \quad {}^2V]\, dA$

$$= \iint_D \quad [\quad U \ \ V - U \quad V]\, dA = \int_{\partial D} [\partial U/\partial\eta\ V - U\ \partial V/\partial\eta]\, ds.$$ **This last equality is a result of the divergence theorem.** *f*

EXERCISE

(1) In the same sense that the Divergence Theorem generalizes the fundamental theorem of integral calculus, and the Green's First Theorem generalizes integration-by-parts, show that Green's Second Theorem leads to

$$\int_a^b f''(x)\, g(x)\, dx - \int_a^b f(x)\, g''(x)\, dx$$

$$= \left[f'(b)\, g(b) - f'(a)\, g(a)\right] - \left[f(b)\, g'(b) - f(a)\, g'(a)\ \right].$$

<u>Hint for 1</u>. Take D to be the rectangle with vertices at [a,0], [b,0], [b,1], and [a,1]. Take U to be [f(x),0] and V to be [g(x),0]. Use Green's second identity. recall that *ds* is arclength.

(2) A. Find a matrix A, a vector B, and a number C such that

$$[\quad + \{4,5\}]\ u = \quad A \quad u + B \quad u + Cu.$$

B. Suppose that D is a region in the plane with a piecewise smooth boundary. Fill in the blank:

$$\iint_D (\quad [\quad + \{4,5\}]\ u)\ dA = \int_{\partial D} [\ \text{blank}\ ].$$

(3) A. Find F such that

$$[3 \frac{\partial^2 u}{\partial x^2} + 5 \frac{\partial^2 u}{\partial y^2}]\, v - u\, [3 \frac{\partial^2 v}{\partial x^2} + 5 \frac{\partial^2 v}{\partial y^2}] = \quad F.$$

<u>Hint</u>. The left side is $\frac{\partial}{\partial x} [3 \frac{\partial u}{\partial x} v - 3\, u \frac{\partial v}{\partial x}] + \frac{\partial}{\partial y} [5 \frac{\partial u}{\partial y} v - 5\, u \frac{\partial v}{\partial y}].$

B. Suppose that D is a region in the plane with a piecewise smooth boundary. Fill in the blank:

$$\iint_D ([3 \frac{\partial^2 u}{\partial x^2} + 5 \frac{\partial^2 u}{\partial y^2}]\, v - u\, [3 \frac{\partial^2 v}{\partial x^2} + 5 \frac{\partial^2 v}{\partial y^2}]) = \int_{\partial D} [\ \text{blank}\ ].$$

<u>Hint.</u> Use the Divergence Theorem.

(4.) A.Suppose that A is a matrix, B is a vector, and C is a number.

Let $L[u] = \quad A \quad u + B \quad u + Cu$

and $M[v] = \quad A \quad v - B \quad v + Cv.$

Find F such that $L[u]\,v - u\,M[v] = F$.