## Section 3.5 ADJOINTS OF DIFFERENTIAL OPERATORS IN TWO DIMENSIONS

In the chapters that came before, an understanding of the adjoints of linear functions was critical in determining when certain linear equations would have a solution....and even in computing the solutions for some cases. It is, then, no surprise that we shall be interested in the computation of adjoints in this setting, too.

In the general inner product space, the adjoint of the linear operator L is defined in terms of the dot product:

$$< L(u) , v > = < u , L^*(v) >.$$

For ordinary differential equations boundary value problems, the dot product came with the problem in a sense: it was an integral over an appropriate interval on which the functions were defined. For partial differential equations with boundary conditions, the dot product will be over the *region* of interest:

$$< f , g > = \iint_D f(x,y)\, g(x,y)\, dx\, dy.$$

For ordinary differential equations, integration-by-parts played a key role in deciding the appropriate boundary conditions to impose so that the formal adjoint would be the real adjoint. Now, the Green's theorems provide the appropriate calculus.

In fact, Green's Second Identity can be used to compute adjoints of the Laplacian. We will see that the Divergence Theorem is useful for the more general second-order, differential operators.

EXAMPLE: Consider the problem

$$^2u = f,$$
$$u(x,0) = u(x,b) = 0$$
$$\partial u/\partial x(0,y) = \partial u/\partial x(a,y) = 0$$

on an interval [0,a] x [0,b] in the plane. This problem invites consideration of the operator L defined on a manifold as given below:

$$L(u) = \ ^2u$$

and $M = \{u: u(x,0) = u(x,b) = 0,\ \partial u/\partial y(0,y) = \partial u/\partial y(a,y) = 0\}.$

The Second Identity presents a natural setting in which the operator L is self-adjoint in the sense that L = L*. Let u and v be in M:

$$\iint_D [v\, L(u) - L(v)\, u]\, dA = \int_{\partial D} [v\, \partial u/\partial \eta - \partial v/\partial \eta\, u]\, ds = 0.$$

This last equality follows because, on the boundary, all of u, $\partial u/\partial \eta$, v, and $\partial v/\partial \eta = 0$.

In order to discuss adjoints of more general second order partial differential equations, let A, B, C, and c be scalar valued functions. Let b be a vector valued function. Let L(u) be given by

$L(u) = A\partial^2u/\partial x^2 + 2B\partial^2u/\partial x\partial y + C\partial^2u/\partial y^2 + < b, \quad u > + cu.$

**DEFINITION:** The <u>FORMAL</u> ADJOINT is given by

$L^*(v) = \partial^2(Av)/\partial x^2 + 2\partial^2(Bv)/\partial x\partial y + \partial^2(Cv)/\partial y^2 - \quad \cdot (bv) + cv.$

Take A, B, and C to be constant. What would it mean to say that L is formally self-adjoint? That L = L* (formally)? Then $<b, \quad u>$ must be $- \quad \cdot bu = < -b, \quad u > - u \quad \cdot b$. Thus, $2 < b, \quad u > = -u ( \quad \cdot b)$ for all u. Since this must hold for all u, it must hold in the special case that $u \equiv 1$, which implies that $\quad \cdot b = 0$. Taking u(x,y) to be x, or to be y gets that each of $b_1$ and $b_2 = 0$. Hence, if L is formally self adjoint, then b = 0.

**EXAMPLES:**

1. Let $L[u] = 3\frac{\partial^2u}{\partial x^2} + 5\frac{\partial^2u}{\partial y^2}$. The formal adjoint of L is L. Note that

$$L[u]\,v - u\,L[v] = \frac{\partial}{\partial x}\Big(3[\frac{\partial u}{\partial x}v - u\frac{\partial v}{\partial x}]\Big) + \frac{\partial}{\partial y}\Big(5[\frac{\partial u}{\partial y}v - u\frac{\partial v}{\partial y}]\Big)$$

$$= \quad \Big(3[\frac{\partial u}{\partial x}v - u\frac{\partial v}{\partial x}],\ 5[\frac{\partial u}{\partial y}v - u\frac{\partial v}{\partial y}]\Big)$$

2. Let $L[u] = 3\frac{\partial^2u}{\partial x^2} + 5\frac{\partial^2u}{\partial y^2} + 7\frac{\partial u}{\partial x} + 11\frac{\partial u}{\partial y} + 13u$. The formal adjoint of L is

$L^*[v] = 3\frac{\partial^2v}{\partial x^2} + 5\frac{\partial^2v}{\partial y^2} - 7\frac{\partial v}{\partial x} - 11\frac{\partial v}{\partial y} + 13v$. Note that

$$L[u]\,v - u\,L^*[v] = \frac{\partial}{\partial x}\Big(3[\frac{\partial u}{\partial x}v - u\frac{\partial v}{\partial x}] + 7uv\Big) + \frac{\partial}{\partial y}\Big(5[\frac{\partial u}{\partial y}v - u\frac{\partial v}{\partial y}] + 11uv\Big)$$

$$= \quad \Big(3[\frac{\partial u}{\partial x}v - u\frac{\partial v}{\partial x}] + 7uv,\ 5[\frac{\partial u}{\partial y}v - u\frac{\partial v}{\partial x}] + 11uv\Big).$$

3. Let $L[u] = e^x\frac{\partial^2u}{\partial x^2} + 5\frac{\partial u}{\partial y} + 3u$. The formal adjoint of L is L* given by

$L^*[v] = e^x\frac{\partial^2v}{\partial x^2} + 2e^x\frac{\partial v}{\partial x} - 5\frac{\partial v}{\partial y} + (e^x+3)u$. Note that

$$L[u]\,v - u\,L^*[v] = \frac{\partial}{\partial x}\Big(e^x v\frac{\partial u}{\partial x} - u\frac{\partial\, e^x v}{\partial x}\Big) + \frac{\partial}{\partial y}\Big(5uv\Big)$$

$$= \quad \Big(e^x v\frac{\partial u}{\partial x} - u\frac{\partial\, e^x v}{\partial x},\ 5uv\Big).$$

**THE CONSTRUCTION OF M***

We now come to the important part of the construction of the <u>real</u> adjoint: how to construct the appropriate adjoint boundary conditions. We

**are given L and M; we have discussed how to construct L*. We now construct M*. To see what is M* in the general case, the divergence theorem is recalled:**

$$\iint_D \nabla\cdot F\,dx\,dy = \int_{\partial D} \langle F, \eta \rangle\,ds.$$

**The hope, then, is to write v L(u) - L*(v) u as $\nabla\cdot F$ for some suitable chosen F.**

**Theorem. If L is a second order differential operator and L* is the formal adjoint, then there is F such that vL(u) - L*(v)u = $\nabla\cdot$ F.**

**Here's how to see that. Note that**

$$(v\,A\partial^2u/\partial x^2 + v\,C\partial^2u/\partial y^2) - (u\,\partial^2(Av)/\partial x^2 + u\,\partial^2(Cv)/\partial y^2)$$
$$= \partial/\partial x(vA\partial u/\partial x - u\partial(Av)/\partial x) + \partial/\partial y\,(vC\partial u/\partial y - u\partial(Cv)/\partial y)$$

$$= \nabla\cdot\{ v(A\partial u/\partial x,\ C\partial u/\partial y) - u(\partial Av/\partial x,\ \partial Cv/\partial y)\}.$$

**Also,** $v \langle b, \nabla u \rangle + u\,\nabla\cdot(bv)$

$$= v\,b_1\,\partial u/\partial x + b_2\,\partial u/\partial y\,v + u\,\partial(b_1v)/\partial x + u\,\partial(b_2v)/\partial y$$

$$= \nabla\cdot(vbu).$$

**COROLLARY:** $\iint_D [vL(u) - uL^*(v)]$

$$= \iint_D \nabla\cdot\{v(A\partial u/\partial x, C\partial u/\partial y) - u(\partial(Av)/\partial x, \partial(Cv)/\partial y) + vbu)$$

$$= \int_{\partial D} [v\{A\partial u/\partial x, C\partial u/\partial y\} - u\{\partial(Av)\partial x, \partial(Cv)/\partial y\} + vbu\}]\cdot\eta\ ds.$$

**EXAMPLES.**

**1(cont). Let L[u] be as in example 1 above for {x,y} in the rectangle D = [0,1]x[0,1] and M = {u: u=0 on ∂D}. Then, according to Example 1, L = L* and**

$$\iint_D [vL(u) - uL^*(v)]\,dA =$$

$$\int_{\partial D} \left\langle \left\{3\left[\frac{\partial u}{\partial x} v - u\frac{\partial v}{\partial x}\right], 5\left[\frac{\partial u}{\partial y} v - u\frac{\partial v}{\partial y}\right]\right\}, \eta \right\rangle ds.$$

**Recalling that the unit normal to the faces of the rectangle D will be {0.-1}, {1,0}, {0,1}, or {-1,0} and that u = 0 on ∂D, we have that**

$$\iint_D [vL(u) - uL^*(v)]\, dA=$$

$$= - \int_0^1 5\frac{\partial u}{\partial y} v\, dx + 3 \int_0^1 \frac{\partial u}{\partial x} v\, dy$$

$$+ 5 \int_1^0 \frac{\partial u}{\partial y} v\, |dx| - 3 \int_1^0 \frac{\partial u}{\partial y} v\, |dy|.$$

**In order for this integral to be zero for all u in M, it must be that v = 0 on ∂D. And M = M*. Hence, {L, M} is (really) self adjoint.**

**2.(cont) Let L[u] be as in Example 2 above and M = {u: u(x,0) = u(0,y) = 0, and** $\frac{\partial u}{\partial x}(1,y) = \frac{\partial u}{\partial y}(x,1) = 0,\ 0 < x < 1,\ 0 < y < 1\}$**. Using the results from above,**

$$\iint_D [L(u)v - uL^*(v)]\, dA=$$

$$- \int_0^1 5 \frac{\partial u}{\partial y} v\, dx + \int_0^1 [-3\, u \frac{\partial v}{\partial x} + 7uv]\, dy$$

$$+ \int_1^0 [5u \frac{\partial v}{\partial x} + 11\, uv]\, |dx| - \int_1^0 [3 \frac{\partial u}{\partial x} v\, ]\, |dy|.$$

**It follows that** $M^* = \{v: v(x,0) = 0,\ v(1,y) = \frac{3}{7}\frac{\partial v}{\partial x}(1,y),\ v(x,1) = \frac{5}{11}\frac{\partial v}{\partial x}(x,1), \text{ and } v(0,y) = 0\}$.

**3(cont). Let L[u] be as in Example 3 above for [x,y} in the first qaudrant. Let M = {u: u = 0 on ∂D}. Then**

$$\iint_D [vL(u) - uL^*(v)]\, dA=$$

$$=\int_{\partial D} <\{e^x v \frac{\partial u}{\partial x} - u \frac{\partial\, e^x v}{\partial x},\ 5uv\},\ \eta > ds = - \int_0^\infty v \frac{\partial u}{\partial x}\, dy.$$

**Thus, M* = {v: v(0,y) = 0 for y > 0}.**

**EXERCISE**

**1. Suppose that** $L(u) = \partial^2 u/\partial x^2 - \partial^2 u/\partial t^2$ **restricted to M = {u: u(0,t) =u(a,t) = 0 and u(x,0) = ∂u/∂t(x,0) = 0}. Classify L as parabolic, hyperbolic, or elliptic.**

**Find L* . Find F such that v L[u] - L*[v} u = F. What is M*?**

**2. Suppose that L[u] = $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 - \partial^2 u/\partial z^2$ restricted to M = {u: u(0,y,z) = 0, u(1,y,z) = 0, u(x,y,0) = u(x,y,1), ∂u/∂z (x,y,0) = ∂u/∂z(x,y,1), ∂u/∂y(x,0,z) = 3 u(x,0,z) ∂u/∂y(x,1,z) = 5 u(x,1,z)} Classify L as parabolic, hyperbolic, or elliptic. Give L*. Find F such that v L[u} - L*[v] u = F. What is M*?**