SECTION 3.6 THE CONSTRUCTION OF THE GREEN'S FUNCTION FOR THE TWO DIMENSIONAL OPERATOR WITH DIRICHLET BOUNDARY CONDITIONS

We now will construct a Green's function for the following problem: Find u such that

$$^2u = f \text{ on } D, \quad u = g \text{ on } \partial D.$$

In order to simplify notation, we will let P or Q represent points in the plane. For example, P might represent {x,y} and Q represent {a,b}. We indicate that

$$^2G(P,Q) = \partial^2 G/\partial x^2 + \partial^2 G/\partial y^2,$$

instead of partials with respect to a and b, by writing $\;_p{}^2G(P,Q)$.

The function G is to be constructed to have these properties:

$$_p{}^2\, G(P,Q) = \delta(P,Q) \qquad \text{and} \qquad G(\;\,,Q) = 0 \text{ on } \partial D.$$

Having such a G, the following applications of the Green's identities show that we can determine the solution to the partial differential equation:

$$\iint G(P,Q)\, f(P)\, d_pA = \iint G(P,Q) \quad _p{}^2u(P)\;\; d_pA$$

$$= \iint \quad _p\; [G(P,Q) \quad _pu - \quad _pG(P,Q)\, u]\, d_pA + \iint \quad _p{}^2G(P,Q)\, u(P)\, d_pA$$

$$= -\int \partial G/\partial \eta(P,Q)\, u(P)\, d_ps \;+\; \iint \delta(P,Q)\, u(P)\, d_pA$$

$$= -\int \partial G/\partial \eta(P,Q)\, g(P)\, d_ps \;+\; \iint \delta(P,Q)\, u(P)\, d_pA.$$

Thus, having such a G and knowing f and g, we have a formula for u which provides a solution to the problem:

$$u(Q) = \iint G(P,Q)\, f(P)\, d_pA + \int \partial G/\partial \eta(P,Q)\, g(P)\, d_ps.$$

How is such a G constructed? We will do it in two pieces. We construct G = F + R where F is the *fundamental* or *singular* part and satisfies

$$^2\, F = \delta \text{ on } D \text{ and}$$

$$F(\;\,,Q) \text{ is independent of } \Theta \text{ (in polar coordinates).} \quad (*)$$

R is the *regular* part and satisfies

$$^2\, R = 0 \text{ on } D$$

$$R = -F \text{ on } \partial D. \qquad\qquad (**)$$

THE FUNDAMENTAL PART

We begin by constructing F. Recall the formula for $\ ^2$ in polar coordinates:

$$ {}^2u = \frac{1}{r}\,\frac{\partial(r\partial u/\partial r)}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2 u}{\partial \Theta^2}. $$

In seeking F such that $\ ^2F = \delta$, we recall that, in the sense of distributions, $\delta(\{x,y\},\{a,b\}) = 0$ if $\{x,y\} \neq \{a,b\}$. Also, δ is radially symmetric. Thus,

$$ 0 = \ ^2F = 1/r\ \partial(r\partial F/\partial r)/\partial r + 1/r^2\ \partial^2F/\partial\Theta^2 $$
$$ = 1/r\ \partial(r\ \partial F/\partial r)/\partial r. $$

This last equality is because F is independent of Θ. Thus $r\ \partial F/\partial r$ is constant and $F(r) = A\ln(r) + B$ for some A and B. It remains to find A and B. To this point we have not used information about F at {a,b}, only at [x,y} different from [a,b]. Information about F at {a,b} comes through the integral. Integrate over any disk with radius $c > 0$

$$ 1 = \iint\ ^2F\,dA = \iint 1\ \ ^2F\,dA = \iint\ ^21\cdot F\,dA + \int [1\partial F/\partial\eta - \partial 1/\partial\eta\ F]\,ds $$
$$ = \int 1\cdot A/r\,ds = 2\pi A. $$

Thus, $A = 1/2\pi$ and B is undetermined.

$$ F(\{x,y\},\{a,b\}) = \ln[\,(x-a)^2 + (y-b)^2]/4\pi. $$

Now we seek R.

THE REGULAR PART: METHODS OF CONFORMAL MAPPING

Recall this complex arithmetic:

$$ z = x+iy = (x^2+y^2)^{1/2}\exp(\,i\arctan(y/x)\,) $$
$$ = |z|\ \exp(\,i\arg(z)\,), $$

$$ \ln(z) = \ln(|z|) + i\arg(z),\ \text{and}\ \mathrm{Re}\ln(z) = \ln(|z|). $$

Let D be a simply connected region different from the entire plane and let α be a point of D. Let $w_\alpha(z)$ be an analytic function from D into the unit disk $D_1(0)$ for which $w_\alpha'(z) \neq 0$ for any z and for which $w_\alpha(\alpha) = 0$. Then

$$ w_\alpha(z) = (z-\alpha)\,P(z) $$

where P is analytic on D and not zero. Define

$$ G(z,\alpha) = \ln(\,|w_\alpha(z)|\,)/2\pi = \frac{1}{2\pi}\,[\ln(\,|z-\alpha|\,) + \mathrm{Re}\ln(\,P(z))]. $$

We show that this G satisfies (•) and (••) above. In fact, F((x,y),(a,b))

$= \ln(\sqrt{[(x-a)^2+(y-b)^2]}\ /2\pi$. To see that Re $\ln(P(z))/2\pi$ satisfies $\ ^2H = 0$, recall that $f = u + iv$ is analytic if and only if the Cauchy-Riemann equations hold: $\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$. Thus, $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$. For this reason $R(z) = \text{Re}\ln(P(z))/2\pi$ satisfies $\ ^2R = 0$. Also , $w_\alpha(z) = 1$ on ∂D , so that $G(z,\alpha) = 0$ for z on ∂D.

EXAMPLE: A function that maps the disk with radius R onto the unit disk taking α to zero is

$$w_\alpha(z) = R\ (z-\alpha)/(R^2-\alpha^* z)$$

(Reference: Churchill, p83, 3rd edition.). Thus, $G(z,\alpha) = \ln(|w_\alpha(z)|)/2\pi$. In order to do the calculus of $\ ^2u$, we change G from being a function from CxC to a function from $R^2 \times R^2$ into R. Typically, u for the disk is given by polar coordinates: $u(r,\theta)$. To make this change, let $z = r\,e^{i\theta}$ and $\alpha = \rho\,e^{i\varphi}$. We change $|w_\alpha(z)|$ to polar coordinates.First, the top:

$$|R\,(z-\alpha)|^2 = |R(re^{i\theta} - \rho e^{i\varphi})|^2 = |R\rho - Rr\,e^{i(\varphi-\theta)}|^2$$

$$= |R\rho - Rr(\cos(\varphi-\theta) + i\sin(\varphi-\theta))|^2$$

$$= [(R\rho - R\rho(\cos(\varphi-\theta))]^2 + R^2r^2\sin^2(\varphi-\theta)$$

$$= R^2\rho^2 - 2R^2r\rho\cos(\varphi-\theta) + R^2r^2$$

$$R^2(r^2 + \rho^2 - 2r\rho\cos(\theta-\varphi)).$$

The bottom is $|R^2 - \alpha^*\,z|^2$

$$= |R^2 - r\rho\ e^{i(\theta-\varphi)}|^2$$

$$= |R^2 - r\rho(\cos(\theta-\varphi) + i\sin(\theta-\varphi))|^2$$

$$= (R^2 - r\,\rho\cos(\theta-\varphi))^2 + r^2\rho^2\ \sin^2(\theta-\varphi)$$

$$= R^4 + r^2\rho^2 - 2R^2\,r\rho\cos(\theta-\varphi).$$

Thus,

$$|w_\alpha(z))|^2 = \frac{R^2(r^2+\rho^2-2\rho r\cos(\theta-\varphi))}{\rho^2[(R^2/\rho)^2+r^2-2rR^2/\rho\cos(\theta-\varphi)]}$$

and

$$G(r,\theta,\rho,\varphi) = \ln(|W_\alpha(z)|)/2\pi$$

$$= \frac{1}{2\pi} \ln(R/\rho) + \frac{1}{4\pi} \ln(r^2 + \rho^2 - 2r\rho\cos(\theta-\varphi)) -$$

$$\frac{1}{4\pi} \ln((R^2/\rho)^2 + r^2 - 2\, r\, R^2/\rho \cos(\theta-\varphi)).$$

We note the values on the boundary, where $\rho = R$, in this last formulation:

$$G(R, \theta, \rho, \varphi) = \ln(1)/2\pi = 0,$$

and

$$\frac{\partial G}{\partial \eta} = \frac{\partial G}{\partial r} = 0 + \frac{1}{4\Pi} \frac{2r-2\rho \cos(\theta-\varphi)}{r^2+\rho^2 - 2\, r\, \rho \cos(\theta-\varphi)} \Bigg|_{r=R}$$

$$- \frac{1}{4\Pi} \frac{2r - 2R^2/\rho \cos(\theta-\varphi)}{(R^2/\rho)^2 + r^2 - 2rR^2/\rho \cos(\theta-\varphi)} \Bigg|_{r=R}$$

$$= \frac{1}{4\Pi} \frac{2R - 2\, \rho \cos(\theta-\varphi)}{R^2 + \rho^2 - 2\, R\, \rho \cos(\theta-\varphi)}$$

$$- \frac{1}{4\Pi} \frac{2R - 2\, R^2/\rho \cos(\theta-\varphi)}{R^4/\rho^2 + R^2 - 2\, R^3/\rho \cos(\theta-\varphi)}$$

$$= \frac{1}{4\Pi} \frac{2R - 2\rho \cos(\theta-\varphi) - 2\rho^2/R + 2\rho \cos(\theta-\varphi)}{R^2 + \rho^2 - 2R\rho \cos(\theta-\varphi)}$$

$$= \frac{1}{2\Pi} \frac{R^2 - \rho^2}{R(R^2 + \rho^2 - 2R\rho \cos(\theta-\varphi))}$$

This last equality gives the familiar Poisson formula:
The solution for

$$^2u = 0 \quad \text{on } D_R(0)$$
$$u = g \quad \text{on } \partial D$$

is

$$u(\rho,\varphi) = \frac{1}{2\pi} \int (R^2-\rho^2)/[\, R^2 + \rho^2 - 2\, R\, \rho \cos(\theta-\varphi)\,] \quad g(R,\theta)\; d\theta.$$

This follows from the equation for $\frac{\partial G}{\partial \eta}$ by remembering that

$$ds = R\, d\Theta,$$

or

$$d\Theta = \frac{ds}{R}.$$

EXAMPLE: We construct the Green's function for the Dirchlet problem in the upper half plane. That is, we give a formula for U such that $\nabla^2 U =$ f(x,y) if y > 0 and u(x,0) = g(x) for all real x.

Consider $w_\alpha(z)$ given by $(z-\alpha)/(z-\alpha^*)$ for z and α in the upper half plane. Two things must be established: $|w_\alpha(z)| < 1$ and, if x is in R, then $|w_\alpha(x)| = 1$. Suppose that $z = x + i\,y$ and $\alpha = a + i\,b$.

$$\frac{|z-\alpha|^2}{|z-\alpha^*|^2} = \frac{(x-a)^2 + (y-b)^2}{(x-a)^2 + (y+b)^2} = \frac{x^2 + y^2 + a^2 + b^2 - 2(ax + yb)}{x^2 + y^2 + a^2 + b^2 - 2(ax - yb)}$$

Note that since y > 0 and b > 0 then -2yb < 2yb and $|z - a| < |z - a^*|$
and $|x-(a+ bi)|^2 = |x - (a- bi)|^2$.

Hence, $G(z,\alpha) = \ln(|W_\alpha(z)|)/2\pi$

$= [\ln(|z-\alpha|) - \ln(|z-\alpha^*|)]/2\pi.$

Or, in R^2 coordinates,

$$G(x,y,a,b) = [\ln((x-a)^2+(y-b)^2) - \ln((x-a)^2+(y+b)^2)]/4\pi.$$

EXAMPLES OF CONFORMAL MAPPINGS

Some examples of conformal mappings from complex variables can be found in the appendix to Churchill's complex variables text.

EXERCISE Give a conformal mapping from the fourth quadrant onto the unit disk.
Ans. Give a sequence of maps and take the composite to get

$$(z^2 + i)/(z^2 - i).$$

Here are two general comments about finding the Green's function for a simple region D.

(a) if β is in the complex unit disk, then $f(u) = (u - \beta)/(1 - u\beta^*)$ maps the unit disk onto the unit disk taking β to 0.

(b) if f takes the region D onto the unit disk then

$$f(z) - f(\alpha)) / (1-f(z) f(\alpha)^*)$$

takes D onto the unit disk and α in D to zero.

Example. The Green's function for the fourth quadrant is given by

$$W_\alpha(z) = \frac{\dfrac{z^2+i}{z^2-i} - \dfrac{\alpha^2+i}{\alpha^2-i}}{1 - \dfrac{z^2+i}{z^2-i}\,\dfrac{(\alpha^2+i)^*}{(\alpha^2-i)^*}}$$

and $G(z,\alpha) = \ln(|w_\alpha(z)|)/2\pi = \frac{1}{2\pi}[\ln(|f(z) - f(\alpha)|) - \ln(|1-f(\alpha)^* f(z)|)]$.

EXERCISE. Find $w_\alpha(z)$ for the disk with center a and radius b.

EXAMPLE: We find u such that $\nabla^2 u = 0$ in the right half plane and $u(0,y) = 1$ if $|y| < 1$ with $u(0,y) = 0$ if $|y| > 1$.

Note that all the parts of the problem are here. There is the domain of the functions which is the right half plane and which is denoted by D. There is the linear operator L which is the Laplacian: $L(u) = \nabla^2 u$. There is the linear equation we wish to solve: $L(u) = 0$. And, there are the boundary conditions which we denote as $u(0,y) = g(y)$ where $g(y) = 0$ or 1 according as to whether $|y| > 1$ or $|y| < 1$. We lay out how to get u in "five easy steps".

Step1. Get a one-to-one map from D to the unit disk:

$$f(z) = (1-z)/(1+z).$$

Step2. Get a one-to-one map from D onto the unit disk which takes the point α in the right half plane to 0: $w_\alpha(z) =$

$$\frac{\dfrac{1-z}{1+z} - \dfrac{1-\alpha}{1+\alpha}}{1 - \dfrac{1-z}{1+z}\,\dfrac{1-\alpha^*}{1+\alpha^*}} = \frac{\alpha - z}{\alpha^* + z}\;\frac{1+\alpha^*}{1+\alpha}$$

Step3. Get $G(z,\alpha) = \ln(|w_\alpha(z)|) = [\ln(|z-\alpha|) - \ln(|z+\alpha^*|)]/2\pi$. Note that G is broken into the fundamental and regular parts.

Step 4.We change G to rectangular coordinates:

$$G(\{x,y\},\{a,b\}) = [\ln((x-a)^2+(y-b)^2) - \ln((x+a)^2+(y-b)^2)]/4\pi.$$

Step 5. Compute $\partial G/\partial\eta$ on the boundary: $\partial G/\partial\eta = -\partial G/\partial x\,|_{(0,y)}$

$$= 1/\pi \quad a/[a^2+(y-b)^2].$$

Finally we are ready to give the formula for u:

$$u(a,b) = \iint_D G(x,y,a,b)\, 0\, dx\, dy + \int_{\partial D} \partial G/\partial\eta(x,y,a,b)\, g(y)\, ds$$

$$= 1/\pi \int_{-1}^{1} a/[a^2+(y-b)^2]\,dy.$$

Remark: The student may be uncomfortable with the direction of this last

integral, thinking that the direction should be taken so that D is on the left. That's partially correct. The student should also remember that ds = -dy so that the sign of the integral is correct as stated.