**Section 3.8: AN EXAMPLE: THE SHAPE OF A DRUM**

**We are now in a position to determine the equation which describes the shape of a drum. Let D be a region in the plane and U(x,y) be the height of a membrane with a prescribed boundary. That is, we assume that the values of U are known on the boundary of D. If we take g to be that function which describes the values of U on the boundary of D, then we have assumed that $U(x,y) = g(x,y)$ for $\{x,y\}$ on $\partial D$. For the interior of D, we assume that the potential energy of the membrane is proportioned to the surface area**

$$E(U) = K \iint_D \sqrt{1 + (\partial U/\partial x)^2 + (\partial U/\partial y)^2}\, dA.$$

**The question is, how can U be chosen to minimize E? Let U be a surface that minimizes energy. Let Φ be any smooth function with $\Phi(x,y) = 0$ on $\partial D$. Then, a possible shape is $U + \varepsilon\Phi$, $\varepsilon > 0$. The potential energy of this surface changes with ε and is given as a function of ε by the equation**

$$E(U+\varepsilon\Phi) = K\iint_D \sqrt{1+[\partial(U+\varepsilon\Phi)/\partial x\}^2 + [\partial(U+\varepsilon\Phi)/\partial y]^2}\ \ dA.$$

**By hypothesis, $E(U) \leq E(U+ \varepsilon\Phi)$. That is,**

$$\frac{\partial E}{\partial \varepsilon}\Big|_{\varepsilon = 0} = 0.$$

**Use the approximation $\sqrt{1+z} \approx 1 + z/2$. Then**

$$E(U) \approx K\iint_D [\, 1 + ((\partial U/\partial x)^2 + (\partial U/\partial y)^2)/2 \,]\, dA$$

**and**

$$E(U+ \varepsilon\Phi) \approx K\iint_D [\, 1 + ((\partial(U+\varepsilon\Phi)/\partial x)^2 + (\partial(U+\varepsilon\Phi)/\partial y)^2)/2 \,]\, dA.$$

**Now, compute dE/dε and evaluate at ε=0. Since E(U) is minimum, this derivative should be zero.**

$$0 = dE/d\varepsilon\,|_{\varepsilon=0} = K/2 \iint_D [\, 2\, \partial U/\partial x\ \ \partial\Phi/\partial x + 2\, \partial U/\partial y\ \ \partial\Phi/\partial y \,]\, dA$$

$$= K\iint_D <\ \ U,\ \ \ \Phi > dA.$$

**Use Green's First Identity to get**

$$dE/d\varepsilon\,|_{\varepsilon=0} = K\int_{\partial D} < \Phi\ \ \ U , \eta > - K \iint_D \Phi\ \ \ {}^2U\, dA$$

**Since $\Phi = 0$ on $\partial D$ then $0 = -K\iint_D \Phi\ \ \ {}^2U\, dA$ for all Φ with $\Phi = 0$ on $\partial D$.**
**Therefore, $\ {}^2U = 0$.**

This result on the shape of a drum shows that a drum at rest, not changing in time, will be situated so that it satisfies a Dirichlet problem. It is resting in the steady state. A drum in the transition state - moving from some initial conditions to the steady state - will satisfy

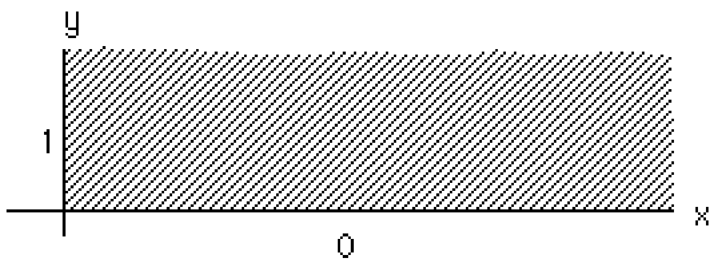
$$\frac{\partial^2 u}{\partial t^2} = {}^2u \quad \text{in } D$$

$$u = g \text{ on } \partial D$$

$$u(0,x,y) = \text{initial distribution on } D,$$

$$\frac{\partial u}{\partial t}(0,x,y) = \text{initial velocity on } D.$$

Other physical situations which arrange themselves inorder to minimize energy will often lead to elliptic problems, too. For example, consider the following heat problem: Suppose the temperature at each point in the upper- right quarter plane has assumed a value u(x,y) such that u has a continuous second partial derivative and the temperature is constant 0 along the positive x axis and constant 1 along the positive y axis. What must be the values of u for other {x,y}? Or, what is the shape of the graph of u?




The mathematical formulation of this problem is as follows: Find u such that

| | | |
|---|---|---|
| | ${}^2u = 0$ | for $x > 0$ and $y > 0$, |
| with | $u(x,0) = 0$ | for $x > 0$, |
| | $u(0,y) = 1$ | for $y > 0$. |

The answer is $u(x,y) = \frac{2}{\pi} \arctan(\frac{y}{x})$. (Exercise 1 below asks you to check this.)

Because the steady state - time independent - heat equation satisfies the same Dirichlet equations, all the results of this and the previous section apply to the heat equation, as well as to the equation for a drum.

We now investigate the uniqueness of solutions for the problem

$$\partial u/\partial t = {}^2u \text{ on } D$$

with $u(x,y,0) = f(x,y)$ on D,

and $u(t,x,y) = g(t,x,y)$ on $\partial D$.

This equation represents the transition of temperature from an initial distribution of g to the steady state with prescribed boundary conditions. The claim is that this problem has no more than one solution.

**THEOREM (UNIQUENESS OF SOLUTION FOR THE TIME DEPENDENT HEAT EQUATION)**

There is only one solution to the equation

$$\partial u/\partial t = \;\; {}^2u \text{ on } D$$

with $u(x,y,0) = f(x,y)$ on D,

and $u(x,y,t) = g(x,y,t)$ on $\partial D$.

Here's a way to see that solutions are unique. Let u and v be solutions and $w = u - v$. Then

$$\partial w/\partial t - \;\; {}^2w = 0 \text{ on } D$$

with $w(x,y,0) = 0$ on $\partial D$,

and $u(x,y,t) = 0$ on $\partial D$.

Consider $E(t) = \iint_D w^2(x,y,t)\,dx\,dy/2$.

We have $E'(t) = \iint_D w(x,y,t)\,\partial w/\partial t(x,y,t)\,dx\,dy$

$$= \iint_D w(x,y,t) \;\; {}^2w(x,y,t)\,dx\,dy$$

$$= \int_{\partial D} < w \;\; w, \eta > ds - \iint_D \| \;\; w \|^2\,dx\,dy.$$

This last comes from Green's First Identity. Recall $w = 0$ on $\partial D$. Then, $E'(t) \le 0$ and E is not increasing. Also $E(t) \ge 0$ and $E(0) = 0$. This means $E(t) = 0$ for all t. *f*

**EXERCISE.**

1. Show that if $u(x,y) = \frac{2}{\pi}\arctan(\frac{y}{x})$ for $x > 0, y > 0$, then

$${}^2u = 0 \quad \text{for } x > 0 \text{ and } y > 0,$$

with

$$u(x,0) = 0 \quad \text{for } x > 0,$$
$$u(0,y) = 1 \quad \text{for } y > 0.$$

2. Show that if $u(r,\theta) = \ln(r)/\ln(2)$ then

$${}^2u = 0 \quad \text{for } 1 < r < 2,\ 0 \le \theta \le 2\pi.$$

$u(1,\theta) = 0$ü
ý for $0 \le \theta \le 2\pi$.
$u(2,\theta) = 1$þ

3. Let $K(t,x) = \exp(-x^2/4t) / \sqrt{4\pi t}$.

a. Show that $\frac{\partial K}{\partial t} = \frac{\partial^2 K}{\partial x^2}$ for $t > 0$ and $-\infty < x < \infty$.

b. Sketch the graph of $K(t,x)$ for $t = 1, \frac{1}{2}, \frac{1}{4}$.

**c. Suppose that f is continuous and bounded for all real numbers and that**

$$u(t,x) = \int_{-\infty}^{\infty} K(\, t \,,\, x-y \,)\, f(y)\, dy \text{ for } t > 0 \text{ and all real } x.$$

**Show that** $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ **[ Using the methods of Laplace transforms, one can show that u(0,x) = f(x). Take MATH 4581 or see page 230 of <u>AN INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS AND HILBERT SPACE METHODS</u> by Karl E. Gustafson.]**

**4. Let** $K(x,y) = \dfrac{1}{\pi}\,\dfrac{y}{x^2 + y^2}$ **.**

**a. Show the** ${}^2K = 0$ **for** $x > 0,\ y > 0$ **and** $K(x,0) = 0$ **for** $x \neq 0$.

**b. Sketch the graph of K(x,y) for** $y = 1, \frac{1}{2}, \frac{1}{4}$.

**c. Suppose that f is continuous and bounded for all real numbers and that**

$$u(x,y) = \int_{-\infty}^{\infty} K(\, x-s \,,\, y \,)\, f(s)\, ds \text{ for } x > 0,\ y > 0.$$

**Show that** ${}^2u = 0$ **for** $x > 0,\ y > 0$. **[We will show that u(x,0) = f(x). See also page 128 of the book cited above or Chapter 11 of Churchill & Brown.]**