Preface

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The Second Alternative

AN INTRODUCTION TO THE PROBLEMS OF GREEN'S FUNCTIONS

The following three problems illustrate, in a simple way, the primary concern of this course.

SAMPLE PROBLEM 1: Let $B = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$

Suppose v is a vector in R^2**. If u is a vector then**

(a) $\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} u = v$

if and only if

(b) v is a vector and u = Bv.

The equivalence of these two is easy to establish. Even more, given only statement (a), you should be able to construct B such that statement (b) is equivalent to statement (a).

SAMPLE PROBLEM 2: Let K(x,t) = 1 + x t. The function u is a solution for

(a) $u(x) = \int_0^1 K(x,t)u(t)\,dt + x^2$ **for x in [0,1]**

if and only if

(b) $u(x) = x^2 - (25+12x)/18$ **for x in [0,1].**

If one supposes that u is as in (b), then the integral calculus should show that u satisfies (a). On the other hand, the task of deriving a formula for u from the relationship in (a) involves techniques which we will discuss in this course.

SAMPLE PROBLEM 3: Let

$$K(x,t) = \begin{cases} x\,(1-t) \text{ if } x \le t \\ t\,(1-x) \text{ if } t \le x. \end{cases}$$

Suppose f is continuous on [0,1]. The function g is a solution for

(a) $g'' = -f$ **and** $g(0) = g(1) = 0$

if and only if

(b) $g(x) = \int_0^1 K(x,t)f(t)dt.$

VERIFICATION OF SAMPLE PROBLEM 3.

(a) (b) Suppose that f is continuous on [0,1] and g'' = −f with

$$g(0) = g(1) = 0.$$

Suppose also that K is as given by sample problem (3). Then

$$\int_0^1 K(x,t)f(t)dt = -\int_0^1 K(x,t)g''(t)dt$$

$$= -(1-x)\int_0^x t\, g''(t)dt - x\int_x^1 (1-t)\, g''(t)dt.$$

Using integration by parts this last line can be rewritten as

$$-(1-x)\left([x\, g'(x) - 0\, g'(0)] - \int_0^x g'(t)dt\right)$$

$$-x\left([(1-1)g'(1) - (1-x)g'(x)] + \int_x^1 g'(t)dt\right)$$

$$= -(1-x)[x\, g'(x) - (g(x)-g(0)\}$$

$$-x[-(1-x)\, g'(x) + (g(1) - g(x))]$$

$$= (1-x)\, g(x) + x\, g(x) = g(x).$$

To get the last line we used the assumption that g(1) = g(0) = 0.

(b) (a) Again, suppose that f is continuous and, now, suppose that

$$g(x) = \int_0^1 K(x,t)\, f(t)\, dt.$$

Then, $$g(0) = \int_0^1 K(0,t)\, f(t)\, dt = 0,$$

$$g(1) = \int_0^1 K(1,t)\, f(t)\, dt = 0, \text{ and}$$

$$g(x) = (1-x)\int_0^x t\, f(t)\, dt + x\int_x^1 (1-t)\, f(t)\, dt,$$

$$g'(x) = -\int_0^x t\, f(t)\, dt + \int_x^1 (1-t)\, f(t)\, dt,$$

$$g''(x) = -x\, f(x) - (1-x)\, f(x) = -f(x).$$

As you can see, it is not hard to show that these two statements are equivalent. Before the course is over then, given statement (a), you should be able to construct K such that statement (b) is equivalent to statement (a). Perhaps you can do this already.

SAMPLE PROBLEM 4: Let C be the upper half-plane consisting of points {x,y} such that $y \geq 0$. There is a function u such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

and

$$u(x,0) = \sin(x).$$

In fact, take $u(x,y) = e^{-y} \sin(x)$ for $y \geq 0$ and all x. That this u satisfies the equation can be verified by simple calculus.

It gives insight into the unifying ideas of this course to realize each of these sample problems as being concerned with an equation of the form Lu=v. It is a worthwhile exercise to reformulate each of the problems in this form.

EXERCISE.

1. Let $A = \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix}$. Find B such that, if v is in R^2, then these are equivalent:
 (a) u is a vector and Au = v, and
 (b) v is a vector and u = Bv.

2. Let K be as in SAMPLE PROBLEM 2. Show that if

$$u(x) = 3x^2 - (25 + 12x)/6$$

then u solves the equation

$$u(x) = \int_0^1 K(x,t)\, u(t)\, dt + 3x^2.$$

3. Let

$$K(x,t) = \begin{cases} 0 & \text{if } 0 < x < t < 1 \\ e^{t-x} - e^{2(t-x)} & \text{if } 0 < t < x < 1. \end{cases}$$

Suppose that f is continuous on [0,1]. Show these are equivalent:

(a) $y'' + 3y' + 2y = f$, $y(0) = y'(0) = 0$.

(b) $y(x) = \int_0^1 K(x,t) f(t) dt$.

4. Let $u(r,\theta) = r\sin(\theta)$. Show that

$$\frac{1}{r}\frac{\partial\left(r\frac{\partial u}{\partial r}\right)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

with $u(1,\theta) = \sin(\theta)$.