Section 19: Compact Operators and Orthonormal Families

A question arises about how compact operators map orthonormal families. If \( T \) is compact and normal and \( \{ \phi_p \}_{p=1}^\infty \) is the sequence of orthonormal eigenvalues, then \( \lim_{n \to \infty} \{ T \phi_n \} = \lim_{n \to \infty} \{ \phi_n \} = 0 \). What if \( T \) is compact, but not necessarily normal (or self adjoint) and \( \{ \phi_p \}_{p=1}^\infty \) is an orthonormal family. Must \( \lim_{n \to \infty} \{ T \phi_n \} = 0 \)?

**Theorem 39** Suppose that \( T \) is compact and \( \{ \phi_p \}_{p=1}^\infty \) is an orthonormal family. Then \( \lim_{n \to \infty} \{ T \phi_n \} = 0 \).

**Proof:** Suppose not. There is a subsequence \( \{ \phi_{u(p)} \}_{p=1}^\infty \) such that \( |T \phi_{u(p)}| \geq \varepsilon \) for some \( \varepsilon \geq 0 \). Since \( \{ \phi_{u(p)} \}_{p=1}^\infty \) is bounded and \( T \) is compact, there is a subsequence of \( \{ T \phi_{u(p)} \}_{p=1}^\infty \) that converges and has limit, say, \( v \neq 0 \). Then,

\[
\lim_{n \to \infty} \langle T \phi_{u(n)}, v \rangle = \langle v, v \rangle \neq 0.
\]

But, also,

\[
\lim_{n \to \infty} \langle T \phi_{u(n)}, v \rangle = \lim_{n \to \infty} \langle \phi_{u(n)}, T^* v \rangle = 0
\]

because these last are terms in the Fourier expansion of

\[
T^* v = \sum_p \langle T^* v, \phi_p \rangle \phi_p
\]

The terms of this sum must go to zero. This gets a contradiction.

**Remark** The ideas around Theorem 38 provides an easy characterization of when operators commute.

**Theorem 40** Suppose that \( A \) and \( B \) are compact, normal operators. These are equivalent:
(a) \( AB = BA \), and
(b) There is a maximal orthonormal family \( \{ \phi_p \}_{p=1}^\infty \) which are eigenvectors for \( A \) and for \( B \).
Proof. If (b) holds, this is clear from the representation of Theorem 36. Suppose that (a) holds and $\lambda$ is an eigenvalue of $A$. Let $S$ be the subspace of vectors $x$ such that $Ax = \lambda x$. Because $A$ and $B$ commute, $B$ maps $S$ into $S$. Hence there is a sequence of eigenvectors for $B$ that spans $S$. But, each of these is an eigenvector for $A$ corresponding to $\lambda$. This process is symmetric in $A$ and $B$. The representation of Theorem 36 completes the result.

Remarks

(1) if $\lambda$ is a nonzero eigenvalue for $BA$, then it is an eigenvalue for $AB$. To see this, suppose $BAx = \lambda x$. Then

$$(AB)Ax = A\lambda x = \lambda A x.$$  

Thus, $\lambda$ is a nonzero eigenvalue for $AB$.

2) Some use this last result to characterize normal operators this way: an operator is normal if and only if it and its adjoint can be "simultaneously diagonalized."

(3) This course has investigated the representation of linear transformations as $\sum_{p=1}^{\infty} \lambda_p \langle x, \phi_p \rangle \phi_p$. This representation gives insight as to the nature of linear transformations. We have found the representation is appropriate for compact, self-adjoint and normal operators. From examples, we have seen that it gives an understanding to bounded, even if not compact operators, and even to unbounded operators on a Hilbert space. The representation should give insight and unification to some of the ideas that are encountered in a study of integral equation, Green's functions, partial differential equations, and Fourier series.

Assignment

(1) Find the eigenvalues for

$$R(x) = \sum_{p=1}^{\infty} \langle x, \phi_p \rangle \phi_{p+1}$$ and for $L(x) = \sum_{p=1}^{\infty} \langle x, \phi_{p+1} \rangle \phi_p$.

(One of these has no eigenvalue and the other has every number in the unit disk as an eigenvalue.)

(2) Do the weighted left shift and right shift have a representation in the simple paradigm?

Maple Remark: The finite dimensional analogue to the right shift and the left shift might be explored as follows:

```maple
> with(linalg):
> R := array([[0, 0, 0, 0], [1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0]]);
> L := array([[0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1], [0, 0, 0, 0]]);
```
Looked at this way, we see that right-shift and left-shift are infinite dimensional analogues of nilpotent operators. We should check to confirm what are the eigenvalues and eigenvectors of these two.

\[ \text{eigenvects}(R); \text{eigenvects}(L); \]

There is a new idea that should be brought up here. One could define "generalized eigenvectors" for A as vectors v for which there is a number \( \lambda \) such that

\[ (A - \lambda I)v \neq 0 \]

but for which

\[ (A - \lambda I)^2 v = 0. \]

This would be a generalized eigenvector v of rank 2. One could define a generalized eigenvector of rank k.

Here's an example:

\[ A := \text{array}([[1,1,2],[0,1,3],[0,0,2]]) ; \]
\[ \text{charpoly}(A, x) ; \]
\[ \text{eigenvects}(A) ; \]

Note that 1 is an eigenvalue of multiplicity 2, but has only one eigenvector. We go looking for one generalized eigenvector of rank 2. To that end, we find the nullspace of \((A-I)^2\):

\[ A_{\text{Idnty}} := \text{evalm}((A - \text{diag}(1,1,1)) \&^* (A - \text{diag}(1,1,1))) ; \]
\[ \text{nullspace}(A_{\text{Idnty}}) ; \]

Thus, we have that

\[ (A - 2 I)\{5,3,1\} = \{0,0,0\}, \]
\[ (A - 1 I)\{1,0,0\} = \{0,0,0\}. \]

and

\[ (A - 1 I)^2\{0,1,0\} = \{0,0,0\}. \]

Several questions come to mind:

(1) What are the generalized eigenvectors for R and L above?
(2) How do these fit into the context and structure for the paradigm presented in these notes?
Section 20: The Most General Paradigm:  
A Characterization of Compact Operators

The paradigm that has been suggested in these notes is applicable for compact and normal operators. This is a fairly satisfactory state of affairs. Yet, the simple matrices

\[
\begin{pmatrix}
0 & 1 \\
0 & 2
\end{pmatrix}
\text{ and }
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 2
\end{pmatrix}
\]
do not fit into that situation. We will push the representation one more time. In addition to the satisfaction of having a decompositon that is applicable to those two matrices, we will be able to obtain the Fredholm Alternative Theorems for mappings with less hypothesis.

Lemma 41. (1) If \( A \) is bounded and \( B \) is compact then \( AB \) is compact.  
(2) If \( A \) is compact and \( B \) is bounded then \( AB \) is compact.  
(3) If \( T \) is compact, then \( T^* \) is compact.  
(Hint: since \( T \) is compact, then \( TT^* \) is compact and \( \langle TT^*x,x\rangle = |T^*x|^2 \).)

Theorem 42. Suppose that \( T \) is a compact operator from \( E \) to \( E \). There are maximal orthonormal families \( \{\varphi_p\} \) and \( \{\Theta_p\} \) and a non-increasing number sequence \( \{\lambda_p\}_{p=1}^\infty \) such that \( \lim_{p \to \infty} \lambda_p = 0 \), and if \( x \) is in \( E \), then

\[
T x = \sum_{p=1}^\infty \lambda_p \left< x, \varphi_p \right> \Theta_p.
\]
Moreover, the convergence is in the norm of BLT.

Proof: Suppose that \( T \) is compact. First, \( T^* \) is bounded since \( T \) is. To see this,

\[
|T^*x|^2 = \langle T^*x, T^*x \rangle = \langle TT^*x, x \rangle \leq |T||T^*x||x|,
\]
so that \( |T^*x| \leq |T||x| \). Now, knowing that \( T^* \) is bounded and \( T \) is compact, we can get that \( TT^* \) is compact and it is selfadjoint. Moreover, \( \langle T^*Tx, x \rangle \geq 0 \) so that all the eigenvalues of \( TT^* \) are nonnegative. Arrange all the eigenvalues in decreasing order. We have

\[
TT^*x = \sum_p \mu_p \left< x, x_p \right> x_p.
\]
For each \( n \) such that \( \mu_n \neq 0 \), let

\[
y_n = T(x_n)/\sqrt{\mu_n}.
\]
Then \( \langle y_n, y_m \rangle = \langle Tx_n, Tx_m \rangle \sqrt{\mu_n \mu_n} = \langle T^*Tx_n, x_m \rangle \sqrt{\mu_m \mu_n} = \sqrt{\frac{\mu_n}{\mu_m}} \langle x_n, x_m \rangle = 0.\)
Thus, \( \{y_p\} \) is orthogonal, even orthonormal. Extend it to be a maximal.

Then \( T(x_p) = \sqrt{\mu_p} y_p \) even if \( \mu_p = 0 \). Suppose that

\[
x = \sum_p <x, x_p> x_p
\]

\( Tx = T(\sum_p <x, x_p> x_p) = \sum_p <x, x_p> Tx_p = \sum_p \sqrt{\mu_p} <x, x_p> y_p \)

To see that this sum converges in the BLT norm,

\[
| \sum_{n+1} \sqrt{\mu_p} <x, x_p> y_p |^2 = \sum_p \mu_p | <x, x_p> | ^2 \mu_{p+1} | x | ^2
\]

**Assignment**

(20.1) Perhaps you will agree that applying this decomposition to the matrices

\[
\begin{pmatrix}
0 & 1 \\
0 & 2
\end{pmatrix}
\text{ and }
\begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 2
\end{pmatrix}
\]

is irresistible. Note that this is different from the decomposition which we had in the first of these notes.

(20.2) With \( T \) as in Theorem 42, What is \( T^* \)?

**MAPLE Remark:** We will get the generalized paradigm for a matrix \( T \) that is not normal.

\[
A := \text{evalm}(\text{transpose}(T) \cdot T);
\]

\[
\text{eigenvects}(A);
\]

\[
x[1] := \text{vector}([0, 1, 0]); x[2] := \text{vector}([0, 0, 1]); x[3] := \text{vector}([1, 0, 0]);
\]

\[
y[1] := \text{evalm}(T \cdot x[1]/1); y[2] := \text{evalm}(T \cdot x[2]/2); y[3] := \text{evalm}(T \cdot x[3]);
\]

\[
y[3] := (0, 1, 0);
\]

The proposal is that \( T(u) = 1 <u, x_1>y_1 + 2 <u, x_2>y_2 + 0 <u, x_3>y_3 \).

We check this.

\[
u := \text{vector}([a, b, c]);
\]

\[
\text{evalm}(T \cdot u);
\]

\[
\text{evalm}(1*\text{innerprod}(u, x[1])*y[1]
+ 2*\text{innerprod}(u, x[2])*y[2]
+ 0*\text{innerprod}(u, x[3])*y[3]);
\]