Section 24: An Application: A Problem in Control

We will illustrate some of the ideas that we have encountered by considering a control problem. To move gently to this problem, we discuss four examples. The first comes from 2^{nd} year calculus. We work it with 2^{nd} year calculus techniques and then note the disadvantages of those techniques. The problem is re-considered and worked with more powerful methods which lead to a method to solve the control problem.

Example 1 Let M be the line in the intersection of the planes P_1: x+y+z=0 and P_2:x-y+z=0. Also, consider the point {1,2,3} which is not in either plane. The problem is to find the closest point in M to {1,2,3}. Here is a solution using the techniques of calculus:

P_1 is the plane consisting of points {x, y, z} such that

$$<\{x,y,z\}, \{1,1,1\}> = 0.$$

P_2 is the plane consisting of points {x, y, z} such that

$$<\{x,y,z\}, \{1,-1,1\}> = 0.$$

The line M has the same direction as the vector

$\{1,1,1\}\times\{1,-1,1\} = \{2,0,-2\}$ and contains {0,0,0}.

Hence, an equation for this line is M(t) = {2t,0,-2t}. We want to choose t in order to minimize |{2t,0,-2t} - {1,2,3}|

$= (2t-1)^2 + 2^2 + (-2t-3)^2 = D(t).$

Well, $D'(t) = 2(2t-1)2 + 2((-2t-3)(-2) = 16t + 8$. And, $D'(t) = 0$ provided that t = -1/2. Hence, the closest point in M to {1,2,3} is {-1,0,1}.

Remark This method only works in 3-D because the cross product is unique to R^3.

Example 2 Let L_1 and L_2 be linear functions from E to $\mathbb{C}$. Let M be the intersection of the null space of L_1 and of L_2. The problem is: given u in E, find the closest point in E to the null space of L_1 and L_2, By the Riesz theorem, there is y_1 and y_2 in E such that $L_1 = <\ , y_1>$ and $L_2 = <\ , y_2>$. It follows that $M = \{x: <x, y_1> = 0 = <x, y_2> \}$. Then $M = \{x: <x, \alpha y_1 + \beta y_2> = 0$ for all α and $\beta\}$. Also, M   = span$\{y_1, y_2\}$. Now, suppose that u is a point of E and P_m is the orthogonal projection onto M and provides the closest point in M to u. (Note $u - P_m(u) \neq <u, y_1>y_1 + <u, y_2>y_2$ unless y_1 and y_2 are orthonormal.) We have $u - P_m(u)$ M and $P_m(u)$ M. From the first of these

$$u - P_m(u) = \alpha y_1 + \beta y_2 \text{ for some } \alpha \text{ and } \beta$$

and from the second

$$<P_m(u), y_1> = 0 = <P_m(u), y_2>.$$

Hence,

$$<u, y_1> = \alpha <y_1, y_1> + \beta <y_2, y_1>$$

and

$$<u, y_2> = \alpha <y_1, y_2> + \beta <y_2, y_2>,$$

or

$$\begin{pmatrix} \langle u,y_1\rangle & \langle u,y_2\rangle \end{pmatrix} = \begin{pmatrix} \langle y_1,y_1\rangle & \langle y_2,y_1\rangle \\ \langle y_1,y_2\rangle & \langle y_2,y_2\rangle \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

We can solve this system uniquely provided

$$|y_1|^2\,|y_2|^2 - |\langle y_1,y_2\rangle|^2 \neq 0.$$

And, this inequality holds if y_1 and y_2 are linearly independent.

Assignment

(1) Re-work example 1 in this context.

(2) Find the point in the intersection of the subspaces w+x+y+z = 0 and w-x+y-z = 0 that is closest to {0,1,2,3}.

Example 3 **For many applications, we make one further variation in the problem. Let L_1 and L_2 be the linear functions from E to $\mathbb{C}$. Let M = {x: $L_1(x) = A_1$ and $L_2(x) = A_2$}. Let $L_1 = \langle\ , y_1\rangle$ and $L_2 = \langle\ , y_2\rangle$. Now, M is a closed, convex set. If u $\in$ E and if we seek the closest point in M to u then $P_m(u)$ again provides this point. While $P_m(u)$ is a projection, it is not necessarily linear. We want a formula for $P_m(u)$. To get this formula, see that u - $P_m(u)$ is perpendicular to $N(L_1)\cap N(L_2)$. We know that for all y in M, $\langle u-P_m(u), y-P_m(u)\rangle \leq 0$. In fact, if n $\in$ $N(L_1)\cap N(L_2)$ and y = n+$P_m(u)$ with y $\in$ M then $\langle u-P_m(u), n\rangle \leq 0$. Also, if n $\in$ $N(L_1)\cap N(L_2)$ then -n is also so that, $\langle u-P_m(u), n\rangle \geq 0$. Thus, u - $P_m(u)$ is perpendicular to the intersection of these two nullspaces. As in Example 2, this implies that**

$$u - P_m(u) = \alpha\, y_1 + \beta\, y_2.$$

We also have the equations

$$A_1 = L_1(P_m(u)) = \langle P_m(u), y_1\rangle$$

and

$$A_2 = L_2(P_m(u)) = \langle P_m(u), y_2\rangle.$$

Suppose we know u and seek $P_m(u)$. From the above, we see that $u - \alpha y_1 - \beta y_2 = P_m(u)$. Hence,

$$\langle u, y_1\rangle - \alpha\langle y_1, y_1\rangle - \beta\langle y_2, y_1\rangle = A_1,$$

$$\langle u, y_2\rangle - \alpha\langle y_1, y_2\rangle - \beta\langle y_2, y_2\rangle = A_2.$$

Or,

$$\begin{pmatrix} \langle u,y_1\rangle - A_1 \\ \langle u,y_2\rangle - A_2 \end{pmatrix} = \begin{pmatrix} \langle y_1,y_1\rangle & \langle y_2,y_1\rangle \\ \langle y_1,y_2\rangle & \langle y_2,y_2\rangle \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Example 4 **Suppose that $Q_0 \in R^2$ and A is a 2x2 matrix. Suppose that b $\in R^2$ and $Q_1 \in R^2$. We seek v with minimum norm such that if**

$$Z' = AZ + bv,\ Z(0) = Q_0,$$

then also, $Z(1) = Q_1$. We know

$$Z(t) = \exp(tA)Q_0 + \int_0^t \exp((t-s)A)bv(s)\,ds.$$

Since $Q_1 = Z(1)$, We have

$$Q_1 = \exp(A)Q_0 + \int_0^1 \exp((1-s)A)bv(s)\,ds$$

or,
$$Q_1 - \exp(A)Q_0 = \int_0^1 \exp((1-s)A)bv(s)\,ds.$$

Let $\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = Q_1 - \exp(A)Q_0$. Let L_1 and L_2 be defined so that

$$\begin{pmatrix} L_1(v) \\ L_2(v) \end{pmatrix} = \left(\int_0^1 \exp((1-s)A)b\, v(s)\,ds \right).$$

To ask for the point v with minimum norm which satisfies $A_1=L_1(v)$ and $A_2= L_2(v)$, we choose u in the previous problem to be 0. Let v be $P_m(0)$ where P_m is the nonlinear projection onto $\{x: L_1(x) = A_1 \text{ and } L_2(x) = A_2\}$. From the above example,

$$v(s) = 0 - \alpha <\exp((1-s)A)b, \{1,0\}> - \beta<\exp((1-s)A)b, \{0,1\}>$$

where α and β satisfy an equation such as that as that just prior to this example.

Assignment Find $v(s):[0,\pi] \to R$ such that $y'' + y = v$, $y(0) = 0$, $y'(0) = 1$, $y(\pi) = 1$, $y'(\pi) = 0$.

Example This example shows that not every point is accessible. Suppose that P_1 and P_2 are orthogonal projections, $A = \alpha P_1 + \beta P_2$, and that $P_1(b) = 0$. If x is in E and u is in $L^2[0,1]$ then

$$|P_1(x) - \int_0^1 \exp((1-s)A)b\, u(s)\,ds| \geq |P_1(x)|.$$

Here's why:

$$|P_1(x) - (\int_0^1 \exp((1-s)\alpha)P_1bu(s)\,ds + \int_0^1 \exp((1-s)\beta)P_2bu(s)\,ds)|^2 =$$

$$| P_1(x) - \int_0^1 \exp((1-s)\beta)u(s)ds\; P_2b |^2$$

$$= |P_1(x)|^2 + |\int_0^1 \exp((1-s)\beta)u(s)\,ds|^2 \; |P_2(b)|^2$$

$$\geq |P_1(x)|^2.$$

Definition If A is a linear transformation, we denote by T the trajectory T = $\{\exp(sA)b: 0 \le s \le 1\}$ and by Ω. the accessible points $\Omega = \{z: \int_0^1 \exp((1-s)A)b\, u(s)\, ds,\ u \quad L^2[0,1]\}$.

Conjecture: $\Omega \quad = T$.