Addendum for Module 17:

Methods to Obtain Solutions for Non-homogeneous Diffusion Equations

Why this Addendum was made:

Concepts AND Methods.

$$\partial u/\partial t = \partial^2 u/\partial x^2 - c\, u(t,x) + A$$

$u(t,0) = \alpha$ and $u(t,1) = \beta$

with $u(0,x) = f(x)$

We know solutions for

$\partial u/\partial t = \partial^2 u/\partial x^2$

$u(t,0) = 0$ and $u(t,1) = 0$

with $u(0,x) = f(x)$.

$u(t,x) = \Sigma A_n \sin(n \Pi x) \exp(- n^2 \Pi^2 t)$

Check this.

We know solutions for

$$\partial w/\partial t = \partial^2 w/\partial x^2 - c\ w(t,x)$$

$w(t,0) = 0$ and $w(t,1) = 0$

with $w(0,x) = f(x)$.

$$w(t,x) = \exp(- c\ t)\ \Sigma\ A_n \sin(n\ \Pi\ x)\ \exp(- n^2\ \Pi^2\ t)$$

Check this.

We address how to solve

$$\partial u/\partial t = \partial^2 u/\partial x^2 - c\, u(t,x) + A$$

$$u(t,0) = \alpha \text{ and } u(t,1) = \beta$$

with $u(0,x) = f(x)$

There are two (or is it three?) non homogeneous parts.

The Steady State Solution

$$0 = \partial^2 u/\partial x^2 - c\ u(x) + A$$

$u(0) = \alpha$ and $u(1) = \beta$

or

$0 = s'' - c\ s(x) + A$, with $s(0) = \alpha$ and $s(1) = \beta$

This is an second order,
ordinary differential equation.

Assign

$$w(t, x) = u(t, x) - s(x).$$

This w will satisfy

$$\partial w/\partial t = \partial^2 w/\partial x^2 - c\, w(t,x)$$

$$w(t,0) = 0 \text{ and } w(t,1) = 0$$

with $w(0,x) = u(0, x) - s(x)$.

Example 1:

$\partial u/\partial t = \partial^2 u/\partial x^2 - 2\, u(t,x) + 1$

$u(t,0) = 1$ and $u(t,1) = 1$

with $u(0,x) = \cos(2 \Pi x)$

Solve $0 = s'' - 2\, s(x) + 1$, with $s(0) = 1$ and $s(1) = 1$.

I did. I used Maple to solve this equation.

Let $w(t,x) = \exp(-2t) \Sigma A_n \sin(n \Pi x) \exp(-n^2 \Pi^2 t)$

Then $u(t, x) = w(t, x) + s(x)$.

$\cos(2 \Pi x) = u(0, x) = w(0, x) + s(x)$.

This last line will determine the coefficients A_n.

Example 2:

$\partial u/\partial t = \partial^2 u/\partial x^2 - 0.002\ (u(t,x) - 1)$

$u(t,0) = 1$ and $u(t,1) = 1$

with $u(0,x) = 5$.

Solve $0 = s'' - 0.002\ (s(x)-1)$, with
$s(0) = 1$ and $s(1) = 1$.

I did. I used Maple to solve this equation.

Let

$$w(t,x) = \exp(-\ 0.002\ t)\ \Sigma\ A_n \sin(n \Pi x) \exp(-\ n^2 \Pi^2 t)$$

Then $u(t, x) = w(t, x) + s(x)$.

$5 = u(0, x) = w(0, x) + s(x)$.

This last line will determine the coefficients A_n.

There is one final example that you can find in the Maple worksheets. It is put there for a simple, but important reason. The techniques are not so different, but

THE STEADY STATE IS NOT STABLE!

The purpose for this addendum to Module 17 is to emphasize that there is a simple, straight forward method for

The Computation of Solutions for

Non-homogeneous Diffusion Equations.