Module 2: Geometry

Definition: In a vector space, the norm of a vector x, denoted $|x|$, has the following properties:

1. $|x| > 0$,

2. if a is a number then $|a\, x| = |a|\,|x|$, and

3. $|x+y| \leq |x| + |y|$.

There are many norms for R^n . Here are three examples in R^2:

1. The one norm:

   $\|[x, y]\|_1 = |x| + |y|$.

2. The two norm (or the usual norm):

   $\|[x,y]\|_2 = \sqrt{x^2 + y^2}$, and

3. The infinity norm (or the max norm):

   $\|[x,y]\|_\infty = \text{Max}(|x|,|y|)$.

There are similar norms in C([0,1])

1. $|f|_1 = \int_0^1 |f(x)|\, dx$

2. $|f|_2 = \sqrt{\int_0^1 |f(x)|^2\, dx}$

3. $|f|_\infty = \max_{0 \le x \le 1} |f(x)|$.

Definition: In a normed space, the distance between two elements is defined in terms of the norm:

$$d(x, y) = | x-y|.$$

Examples: We find the distance from [1,2] to [4,6] and from sin(x) to cos(x) with each of the norms above. With the usual norm, we have

$$\sqrt{(1-4)^2 + (2-6)^2}$$

and

$$\sqrt{\int_0^1 (\sin(x) - \cos(x))^2\, dx}$$

Definition: The dot product or inner product of vectors x and y, denoted <x, y>, is a number valued function with these properties:

(1) if x is in the space and x is not zero, then
< x, x> is positive.

(2) if x and y are in the space then <x, y> = <y, x>*, where * denotes complex conjugate.

(3) if x, y and z are in the space and a is a number then
< a x + y, z> = a <x, z> + <y, z>.

The usual dot product for R^2 and for C([-1, 1]) are given here:

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2$$

and

$$\langle f, g \rangle = \int_{-1}^{1} f(x)\, g(x)\, dx.$$

Examples: We explain some examples in class. These examples are worked out in the associated Maple worksheet.

Remark: $N(x) = \sqrt{< x, x>}$ has all the properties of a norm and is called the norm associated with the indicated dot product.

Example: We illustrate the importance of dot products by finding the solution for these three equations with three unknowns:

Find a, b, and c so that

$$[1, 2, 3] = a\,[1, 2, 1] + b\,[-2, 1, 0] + c\,[-1, -2, 5].$$

Assignment: See the Maple Worksheet

In this Module 2, we have

1. Defined a norm and given examples in R^2 and in C([0,1]).
2. Defined distance in terms of the norm.
3. Defined a dot product on a vector space.
4. Explained how dot products generate norms, which generate distance functions.