Module 3: Orthogonal Families

Definition: The collection of vectors

$$\{ \Theta_1, \Theta_2, \Theta_3, \ldots \}$$

is called orthogonal if

$$< \Theta_i, \Theta_j > = 0, \text{ whenever } i \neq j.$$

The family is orthonormal if each $|\Theta_i| = 1$.

Examples of orthogonal families.

(1) Standard basis in R^n.

(2) The infinite collection of functions on $C[-\pi, \pi]$

$\{ 1, \cos(x), \cos(2x), \cos(3x), \ldots ,$
$\sin(x), \sin(2x), \sin(3x), \ldots \}.$

(3) The infinite collection of functions on $C[0, \pi]$

$\{ 1, \cos(x), \cos(2x), \cos(3x), \ldots \}.$

(4) The infinite collection of functions on $C[0, \pi]$

$\{ \sin(x), \sin(2x), \sin(3x), \ldots \}.$

Example 2,

{ 1, cos(x), cos(2 x), cos(3 x), ... ,
sin(x), sin(2 x), sin(3 x), ... },

is <u>not</u> orthogonal in $C[0, \pi]$.

For example,

$$\int_0^{\pi} \sin(x)\cos(2x)\,dx = -2/3$$

We ask: why is there an interest in orthogonal functions? Here are some understandings:

(a) If the family $\{ \Theta_1, \Theta_2, \Theta_3, \dots \}$
is orthogonal, then it forms a set of linearly independent vectors.

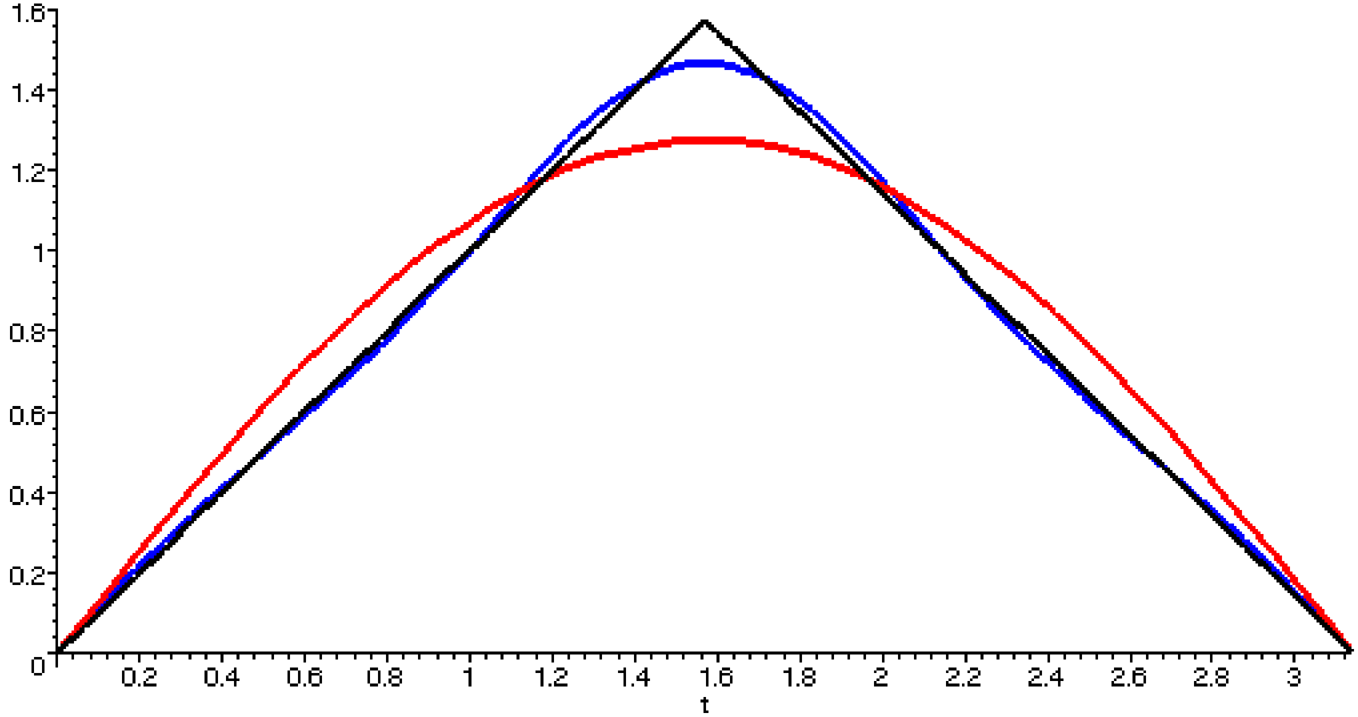
(b) If you tell me that f is a linear combination of $\{\Theta_1, \Theta_2, \Theta_3\}$
-- that is: $f = a\ \Theta_1 + b\ \Theta_2 + c\ \Theta_3$ --
then I can tell you what the coefficients a, b, and c are.

(c) If there are an infinite number of θ's, then finite linear combinations may only approximate the f's.

An illustration follows. Other examples are found in the associated Maple worksheet.

We present a graph with a two term approximation and a five term approximation.

A graph with a two term approximation and a five term approximation.



To answer questions of convergence, we make comparisons with inequalities. Here are three. Proofs for these are found in the references.

CBS: $| < f, g > | \leq |f| \, |g|$.

Triangle: $| f + g | \leq |f| + |g|$.

Fourier equality: With an orthonormal family

$$\left\| f - \sum_p a_p \Phi_p \right\|^2 =$$

$$\| f \|^2 + \sum_p \left(\langle f, \Phi_p \rangle - a_p \right)^2 - \sum_p \left\| \langle f, \Phi_p \rangle \right\|^2$$

$$\left\| f - \sum_p a_p \Phi_p \right\|^2 =$$

$$\|f\|^2 + \sum_p \left(\langle f, \Phi_p \rangle - a_p \right)^2 - \sum_p \left\| \langle f, \Phi_p \rangle \right\|^2$$

Observation: How do you choose the a's so that

$$\left\| f - \sum_p a_p \Phi_p \right\|^2$$

is a small as possible?

$$\left\| f - \sum_p a_p \Phi_p \right\|^2 =$$

$$\| f \|^2 + \sum_p \left(\langle f, \Phi_p \rangle - a_p \right)^2 - \sum_p \left\| \langle f, \Phi_p \rangle \right\|^2$$

With that choice,

Bessel's inequality:

$$\| f \|^2 \geq \sum_p \left| \langle f, \Phi_p \rangle \right|^2$$

$$\left\| f - \sum_p a_p \Phi_p \right\|^2 =$$

$$\| f \|^2 + \sum_p \left(\langle f, \Phi_p \rangle - a_p \right)^2 - \sum_p \left\| \langle f, \Phi_p \rangle \right\|^2$$

If there are "enough" Θ_p 's, then

$$\| f \|^2 = \sum_p \left| \langle f, \Phi_p \rangle \right|^2$$

Assignment: See the Maple Worksheet

In this third module, we

1. defined orthogonal families,
2. illustrated approximations for f with orthogonal families using the important Fourier Coefficient:

$$a_p = \langle f, \Theta_p \rangle \Big/ \|\Theta_p\|^2,$$

3. gave some important inequalities.