Module 4: The Gramm Schmidt Process

Having a set of orthogonal vectors

$\{ \Theta_1, \Theta_2, \Theta_3 \ldots \}$

has become an important idea. These may be converted into an orthonormal set by dividing by the norm:

$$\Phi_n = \Theta_n / \|\Theta_n\|.$$

From the Fourier Inequality, if f is in the space then the element of the span of the Φ_n's that is closest to f is

$$\sum_p \langle f, \Phi_p \rangle \Phi_p \ .$$

This last result is why Fourier Series is a topic for study by scientists and engineers.

Suppose that { $v_1, v_2, v_3, \ldots$ } are linearly independent, but not orthogonal. We create an orthonormal sequence that spans the same set. This is the

GRAMM SCHMIDT PROCESS.

Procedure: The GRAMM SCHMIDT PROCESS

Step 1:

$$\Theta_1 = v_1$$

$$\Phi_1 = \Theta_1 / \|\Theta_1\|$$

Note that $\|\Phi_1\| = 1$.

How do we make the next term?

Step 2:

$$\Theta_2 = v_2 - \langle v_2, \Phi_1 \rangle \Phi_1$$

$$\Phi_2 = \Theta_2 / \|\Theta_2\|$$

Note that

a. $\langle \Phi_1, \Phi_2 \rangle = 0$,

b. $\|\Phi_2\| = 1$.

How is the third one made?

Step 3:

$$\Theta_3 = v_3 - \langle v_3, \Phi_2 \rangle \Phi_2 - \langle v_3, \Phi_1 \rangle \Phi_1,$$
$$\Phi_3 = \Theta_3 / \|\Theta_3\|.$$

Note that

a. $\langle \Phi_3, \Phi_1 \rangle = 0 = \langle \Phi_3, \Phi_2 \rangle$

b. $\|\Phi_3\| = 1.$

The process continues.

Example in R^3.

We perform the Gramm Schmidt Process to the two vectors v_1 = [1, 1, 1] and v_2 = [1, 0, 1].

$\Theta_1 = [1, 1, 1]$, $\Phi_1 = [1, 1, 1]/\sqrt{3}$

$\Theta_2 = [1, -2, 1]/\sqrt{3}$, $\Phi_2 = [1, -2, 1]/\sqrt{6}$.

Check that these are correct. Give the closest point in the plane spanned by

[1, 1, 1] and [1, 0, 1]

to [1, 2, 3].

This process can be done to functions, too. We perform the Gramm Schmidt Process in C([-1,1]) to the functions 1, x, x^2, and x^3.

Step 1:

$$\Theta_1 = v_1 \qquad \Phi_1 = \Theta_1 / \|\Theta_1\|$$

Step 2:

$$\Theta_2 = v_2 - \langle v_2, \Phi_1 \rangle \Phi_1$$

$$\Phi_2 = \Theta_2 / \|\Theta_2\|$$

Step 3:

$$\Theta_3 = v_3 - \langle v_3, \Phi_2 \rangle \Phi_2 - \langle v_3, \Phi_1 \rangle \Phi_1,$$

$$\Phi_3 = \Theta_3 / \|\Theta_3\|.$$

Assignment: See the Maple Worksheet

In this module 4, we have

1. explained the Gramm Schmidt Process,
2. worked out a 3 dimensional example, and
3. referenced a Maple worksheet to work an example in C([-1,1]).