Module 5: Projections

Suppose that E is a vector space with dot product < , > and that $\{\Theta_p\}_{p=1}^{n}$ is an orthogonal sequence.

Let S be the span of the Θ_p 's.

That is,

$$S = \{\ u : u = \sum_{p=1}^{n} \alpha_p \Theta_p\ ,\ \alpha_p \text{ real } \}$$

If f is in E then the Fourier Expansion

$$\sum_{p=1}^{n} \frac{\langle f, \Theta_p \rangle}{\langle \Theta_p, \Theta_p \rangle} \Theta_p$$

is in S and is closer to f than any other element in S.

Choosing the closest point in S generates a projection: $P^2 = P$.

That is: Take f. Form the Fourier Expansion of f. Call what you get P(f). Form the Fourier Expansion of P(f). What you get with this second expansion is P(f) again.

Problem: Obtain the projection onto a plane through the origin.
Solution: Get two linearly independent vectors in the plane: v_1 and v_2.

Perform the Gramm Schmidt process on these two vectors to construct Θ_1 and Θ_2.

Construct the Fourier Expansion for any u:

$$P(U) = \frac{\langle u, \Theta_1 \rangle}{\langle \Theta_1, \Theta_1 \rangle}\Theta_1 + \frac{\langle u, \Theta_2 \rangle}{\langle \Theta_2, \Theta_2 \rangle}\Theta_2.$$

Example:

V1=[1,1,1] and v2=[1,0,1]. U=[1,2,2].

Θ_1 = [1,1,1] and Θ_2 = [1/3,-2/3,1/3]

$P(u) = a_1 \Theta_1 + a_2 \Theta_2$

$= 5/3\ \Theta_1 + (-1/2)\ \Theta_2$

$= [3/2,\ 2,\ 3/2]$

Example: Let

$$F(x) = \begin{cases} 0 \text{ if } x < 1/2 \\ 1 \text{ if } x > 1/2 \end{cases}.$$

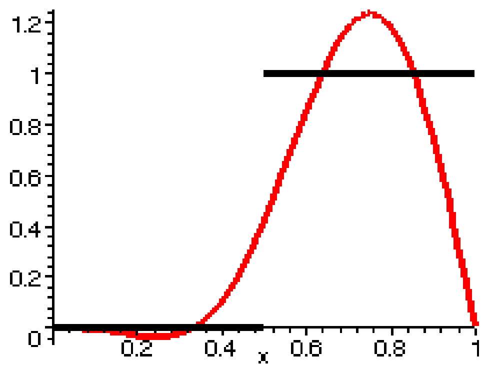
Get the best approximation for f on [0,1] using $\{\sin(x), \sin(2\pi x), \sin(3\pi x), \sin(4\pi x)\}$.

These sine terms are already orthogonal in C([0,1]). We simply compute the coefficients.

$$a_n = \frac{\int_0^1 f(x)\sin(n\pi x)dx}{\int_0^1 \sin(n\pi x)^2 dx}$$

Graph of Heaviside(x-1/2) and

$2/\pi \sin(\pi x) - 2/\pi \sin(2 \pi x) + 2/(3 \pi) \sin(3 \pi x)$



Assignment: See the Maple Worksheet

In this Module 5 we have

1. said what we mean by a projection,
2. explained that the Fourier Expansion with orthogonal vectors is a projection,
3. projected onto a plane, and
4. obtained and graphed the Fourier Expansion for a function.