Module 7: Extensions

Suppose a function is defined on an interval [0, L]. We discuss even, odd, and periodic extensions.

Definition of Even Functions: f is even on the interval [-a, a] if f(x) = f(-x) for all x in the interval.

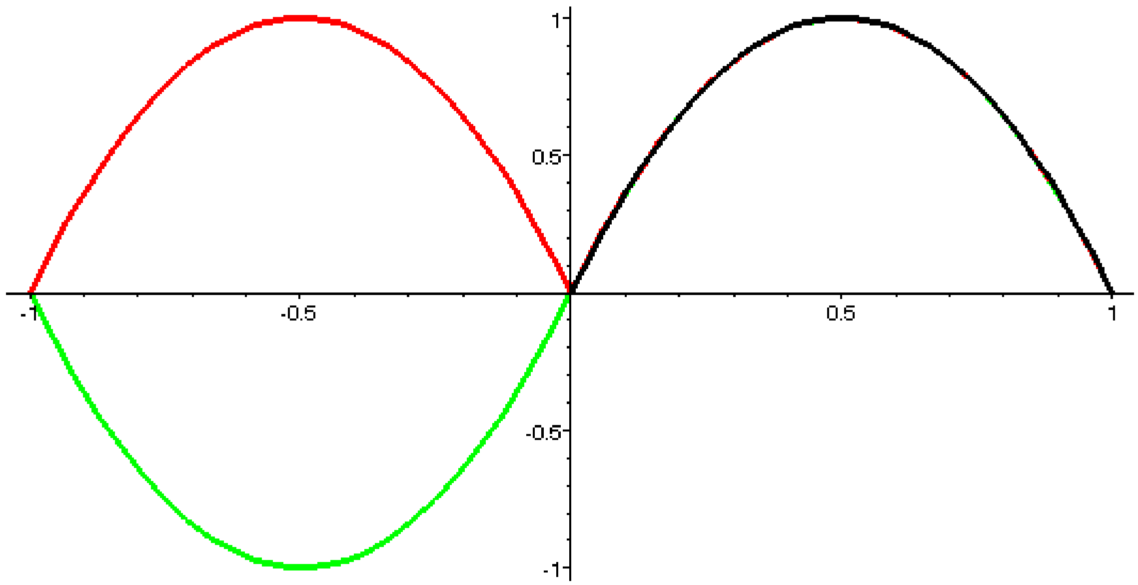
Definition of Odd Functions: f is odd on the interval [-a, a] if f(x) = -f(-x) for all x in the interval.

There is a geometric interpretation.

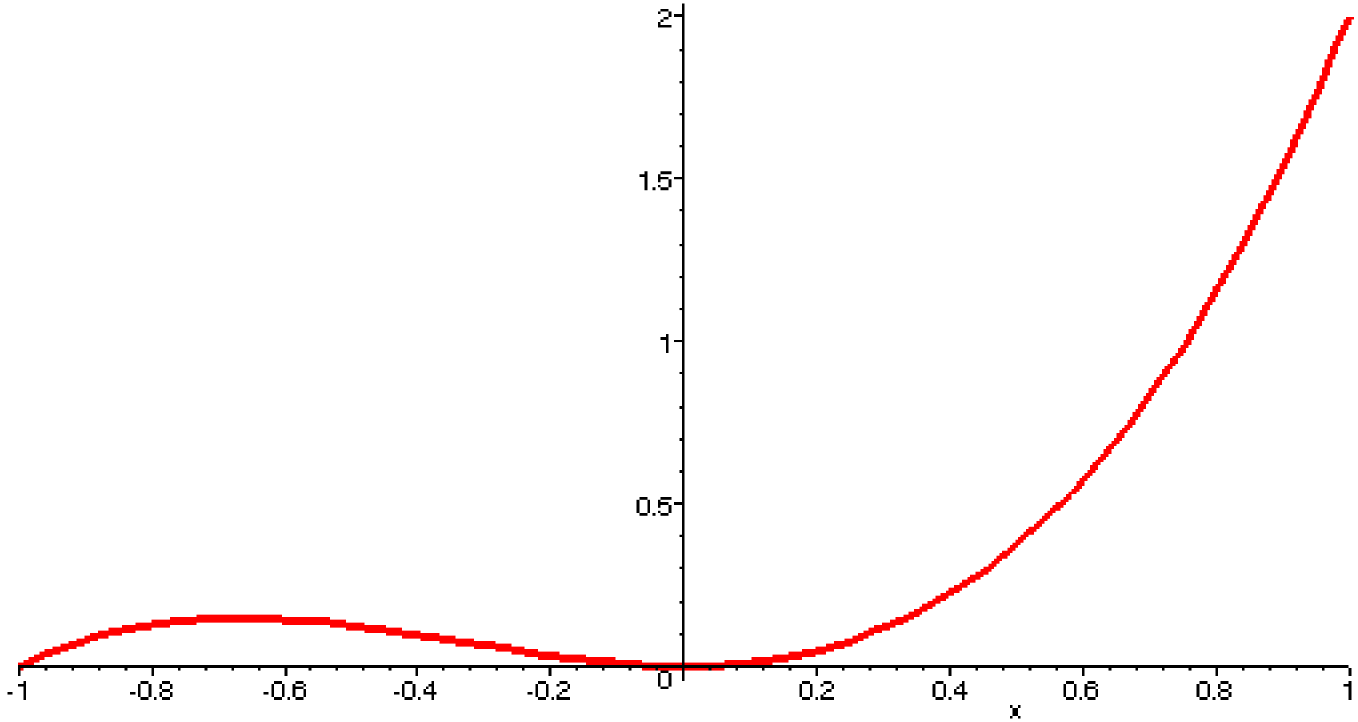
A function is even if its graph is symmetric about the Y axis.

A function is odd if its graph is symmetric about the origin.

Here is the graph of $f(x) = 4x(1-x)$ on the interval $[0, 1]$ and also the even and the odd extension.



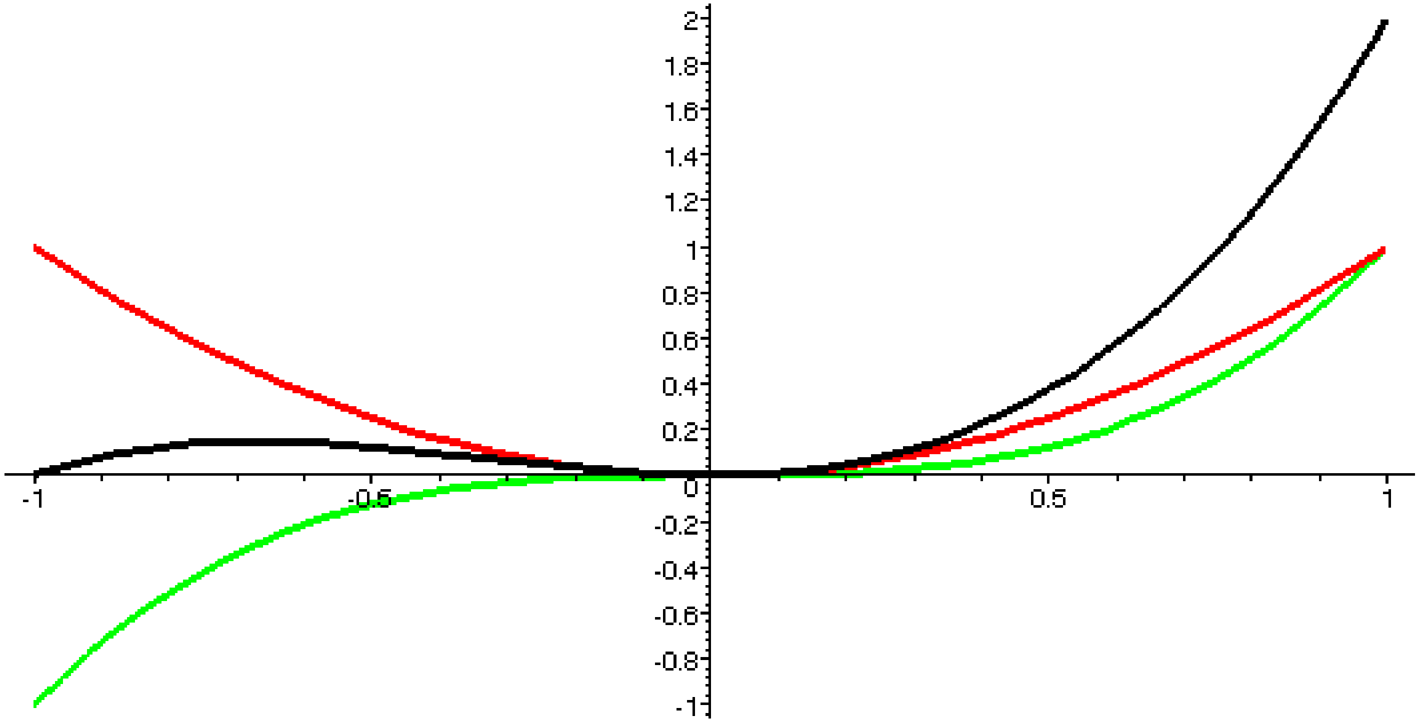
Even and Odd extension of 4x(1-x)



Neither even nor odd

Every function defined on an interval symmetric about the origin can be written as the sum of an even and an odd function.

$$f(x) = [\ f(x) + f(-x)\ ]/2 \ + \ [\ f(x) - f(-x)\ ]/2$$



Even plus Odd = General

Remarks.

(1) Suppose that f is even on the interval [- π, π].

Then $\int_{-\pi}^{\pi} f(x) \sin(nx)dx = 0$.

(2) Suppose that f is odd on the [- π, π]. Then $\int_{-\pi}^{\pi} f(x) \cos(nx)dx = 0$.

(3) From either of the above two remarks it follows that

$$\int_{-\pi}^{\pi} \cos(nx) \sin(nx)dx = 0.$$

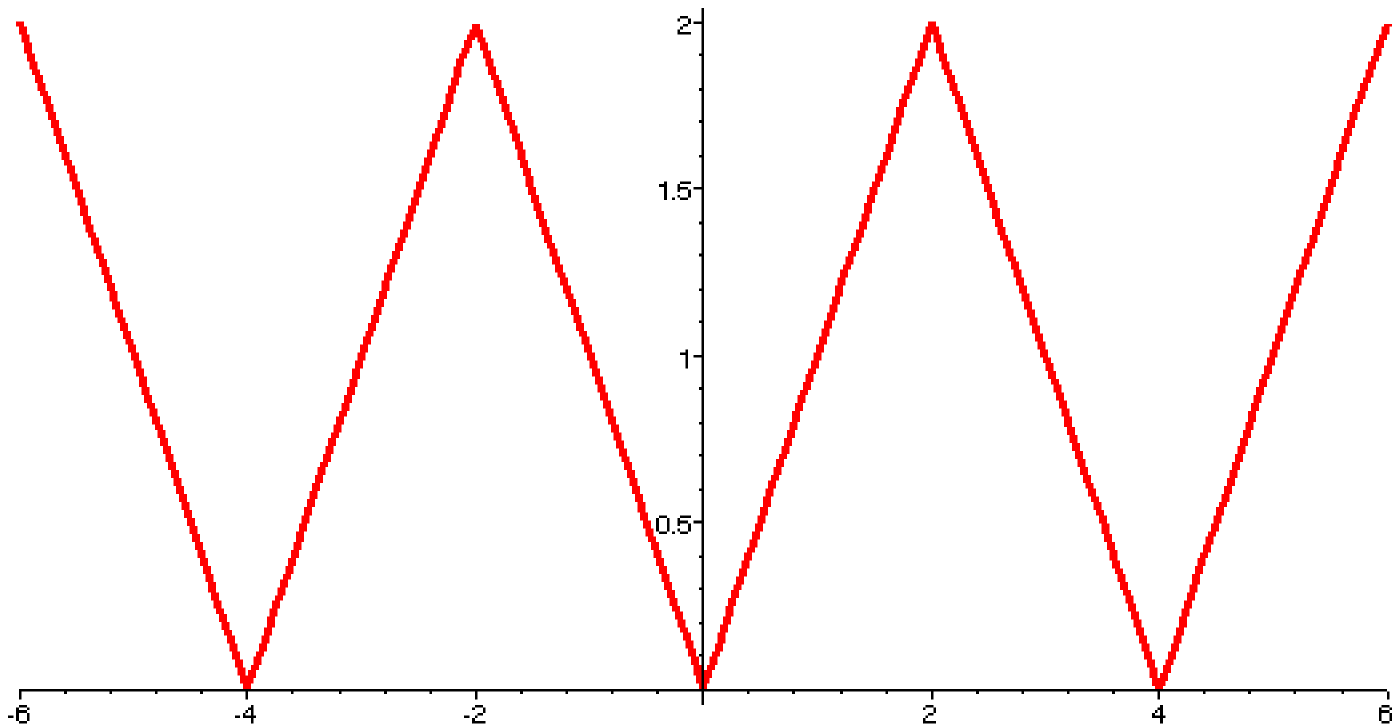
Definition: The function is periodic if there is a number P such that $f(x+P) = f(x)$ for all x. The smallest such positive number P is called the period.

Examples: The sine and cosine functions have period 2π.

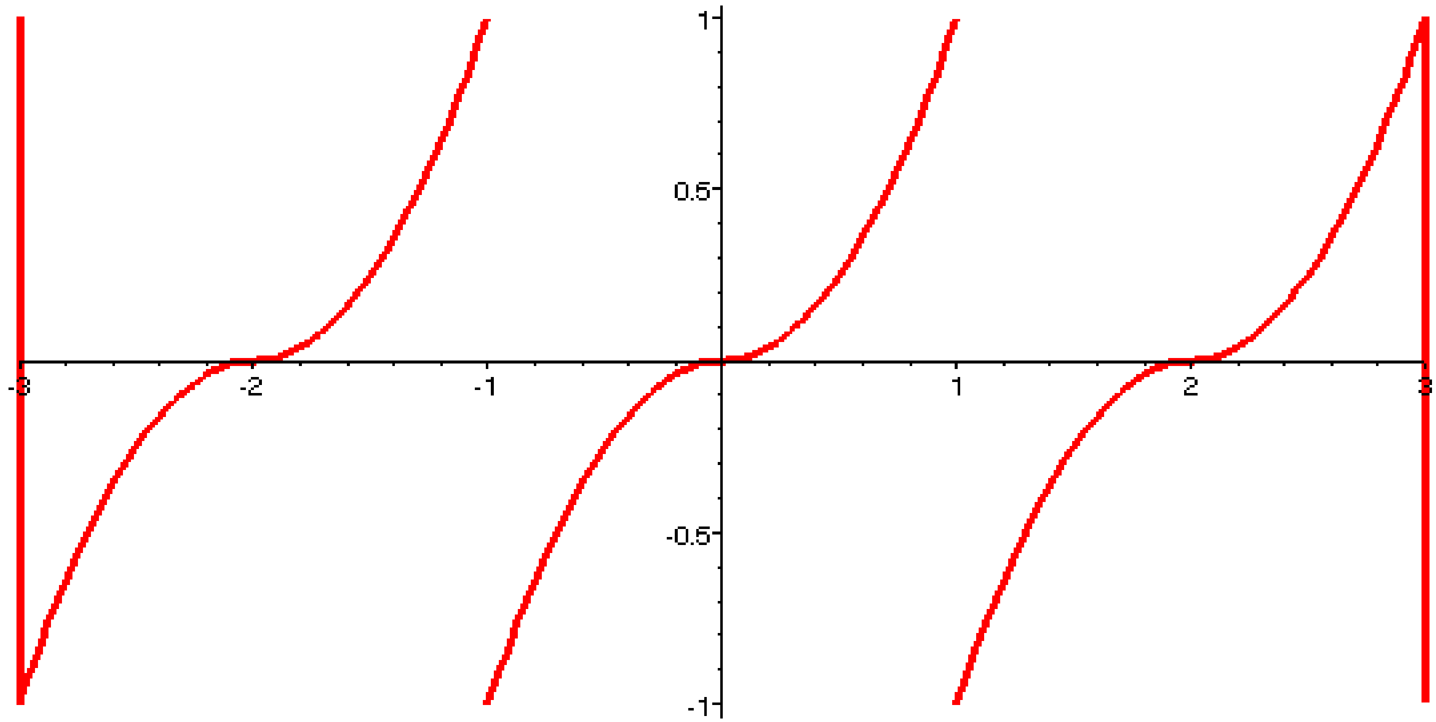
If a function is specified on some interval, then one could ask for a periodic extension.

If the function were defined on an interval [0, L], one could ask for an even periodic extension or an odd, periodic extension. In this case, one would first make an even (or odd) extension, and then make a 2 L periodic extension.

We illustrate with graphs.



Even Periodic Extension



Odd Periodic Extension

1. Here are four possibilities.

a. f is even and has period 2π.

b. f is odd and has period 2π.

c. f has period π.

d. f has period 2π and alternates on each half period, in the sense that

$$f(x+\pi) = -f(x).$$

2. Match these possibilities with the following Fourier Series for f:

a. The series contains only sine terms.

b. The series contains only cosine terms.

c. The series contains sin(nx) and cos(nx) terms, but only for odd values of n.

d. The series contains sin(nx) and cos(nx) terms, but only for even values of n.

Assignment: See the Maple notes.

In this Module 7, we have examined extensions:

1. even extensions
2. odd extensions,
3. periodic extensions, and
4. combinations of these.