Module 9: Convergence for Fourier Series

Three general types of convergence in C([0, 1]):

- normed,
  - pointwise, and
    - uniform.

We apply these ideas to the special case that we have a function f and its associated Fourier series:

$$S_n(x) = \sum_{p=0}^{n} \langle f, \phi_p \rangle \phi_p(x)$$

Definitions:

A function is sectionally continuous on an interval [a, b] if it is continuous on that interval except for possibly a finite number of jumps and removable discontinuities.

A function is sectionally smooth on an interval [a, b] if f and f ' are sectionally continuous on the interval [a, b].

Examples: The function f(x) = signum(x) is sectionally continuous on [-1, 1], but the function g(x) = 1/x is not sectionally continuous on that interval.

Example: The function $\sqrt{|x|}$ is (even) continuous, but not sectionally smooth on [-1, 1] because the derivative goes to infinity as x approaches zero.

THEOREM: If the function f is sectionally smooth and periodic with period 2 c, then at each point x the Fourier series for f converges to

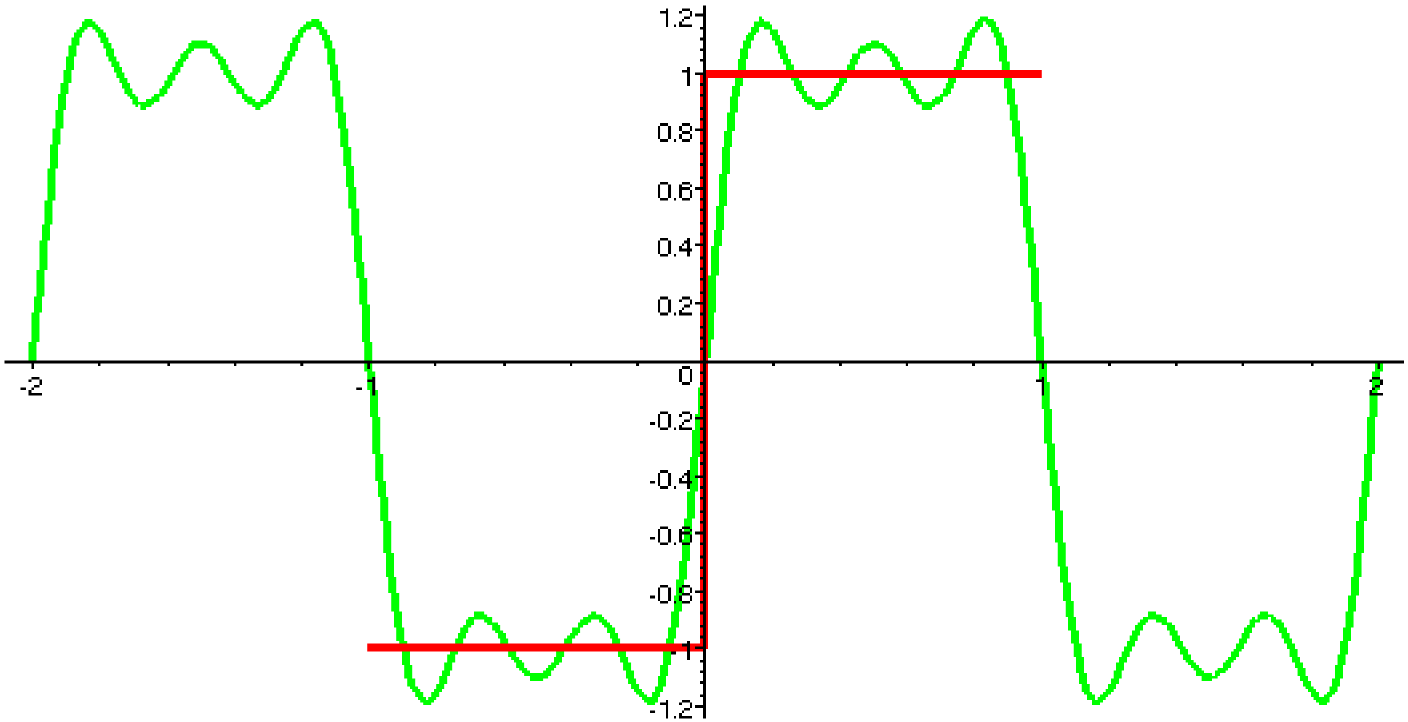
$$[f(x+) + f(x-)] / 2.$$

Example: The function signum(x) is sectionally smooth. Therefore the Fourier series for this function converges

to 1 for $0 < x < 1$,

to -1 for $-1 < x < 0$, and

to 0 for $x = -1, 0,$ or 1.



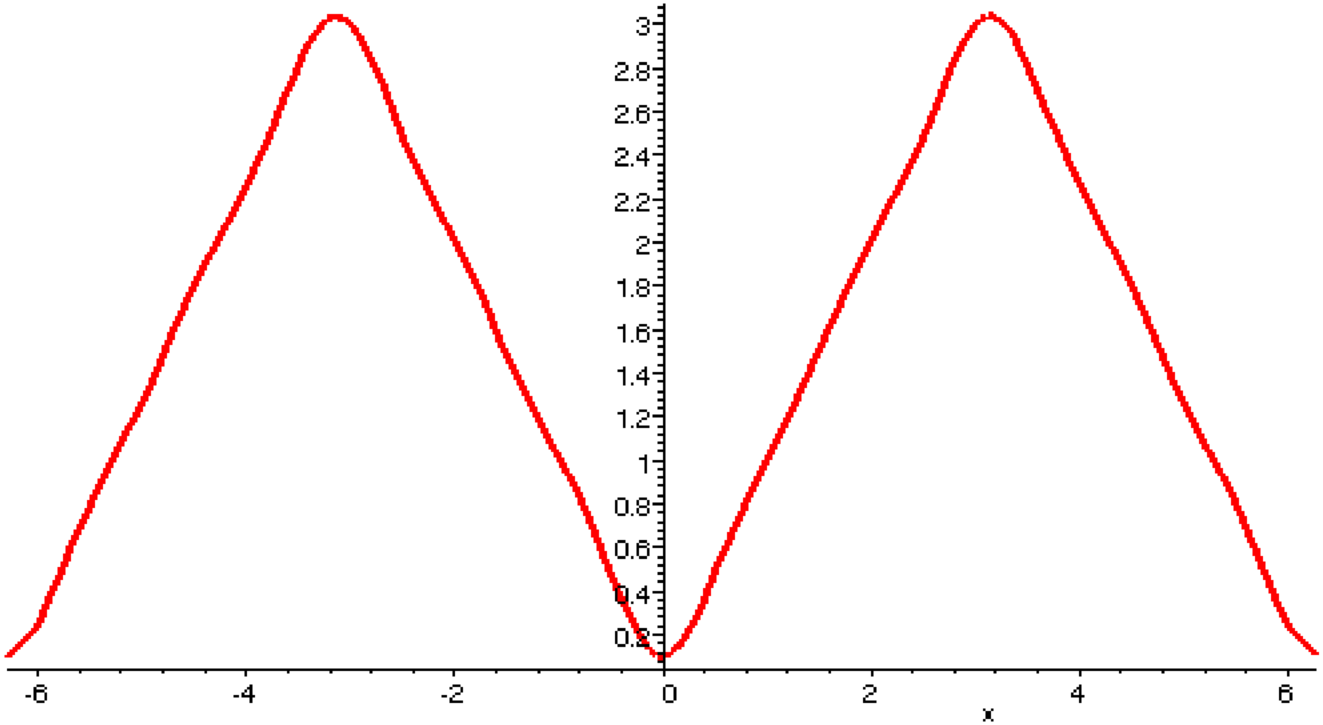
Fourier approximation for signum(x).

THEOREM: If the series $\sum_n |a_n| + |b_n|$ converges, then the Fourier series for f converges uniformly in the interval [-c, c].

Example: Take the series with the b's zero and $\mathbf{a_n} = 4/(n^2\pi)$ for n even and zero of n odd.

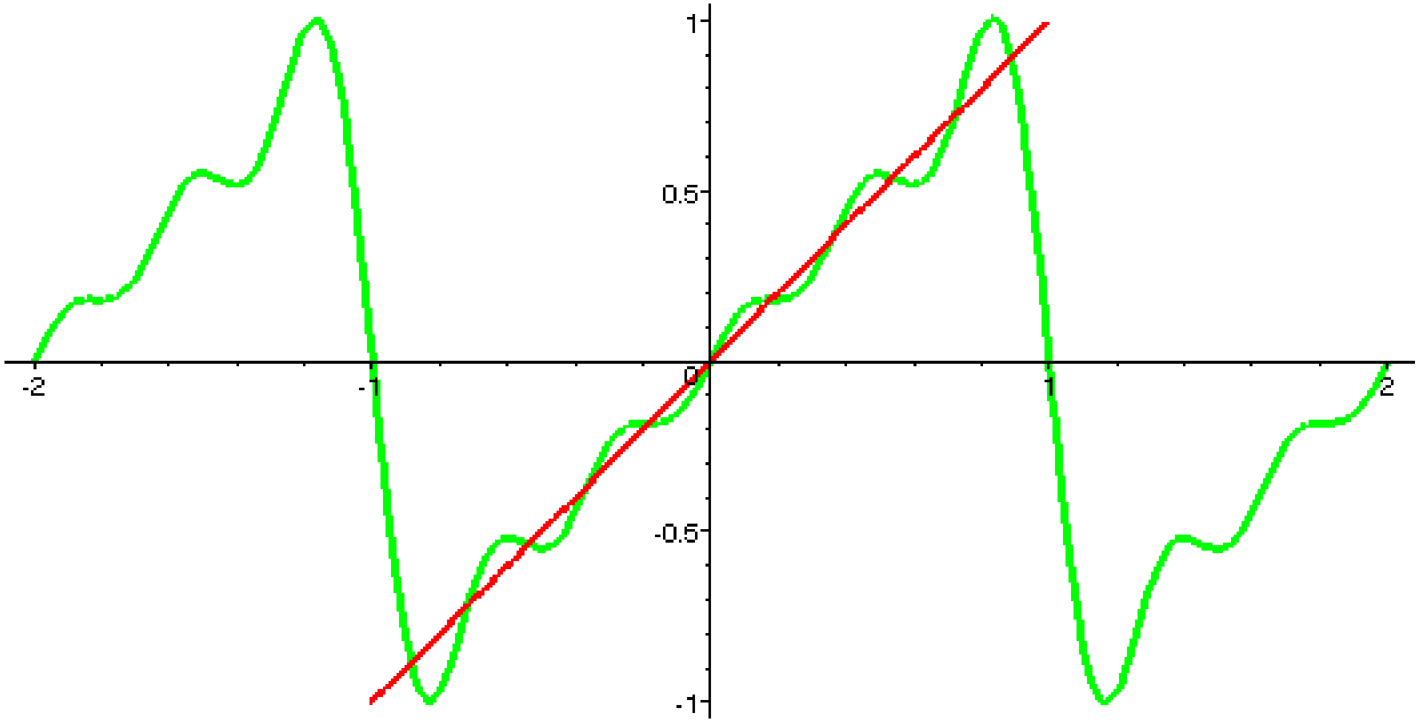
THEOREM: If f(x) is periodic, continuous, and has a sectionally continuous derivative, then the Fourier Series corresponding to f converges uniformly to f(x) for the entire real line.

Take $f(x) = |x|$ on the interval $[-\pi, \pi]$. It is continuous and its periodic extension is continuous. The derivative is sectionally continuous. The coefficients are the ones used in the previous example.



Fourier Approximation for |x|.

The function $f(x) = x$ on the interval $[-1, 1]$ is continuous there. The periodic extension is sectionally continuous, but not continuous.
The Fourier Series does not converge uniformly.



Fourier Approximation for x on [-1, 1].

Assignment: See the Maple Worksheet

In this ninth module we have:
Contrasted the three types of convergence for Fourier Series and provided examples. There were three important Theorems which gave conditions to imply the nature of convergence.