Module 10: Calculus on Fourier Series

We examine the nature of the calculus on series.

We ask if the derivative of a Fourier series representing f is a series which will represent the derivative of f.

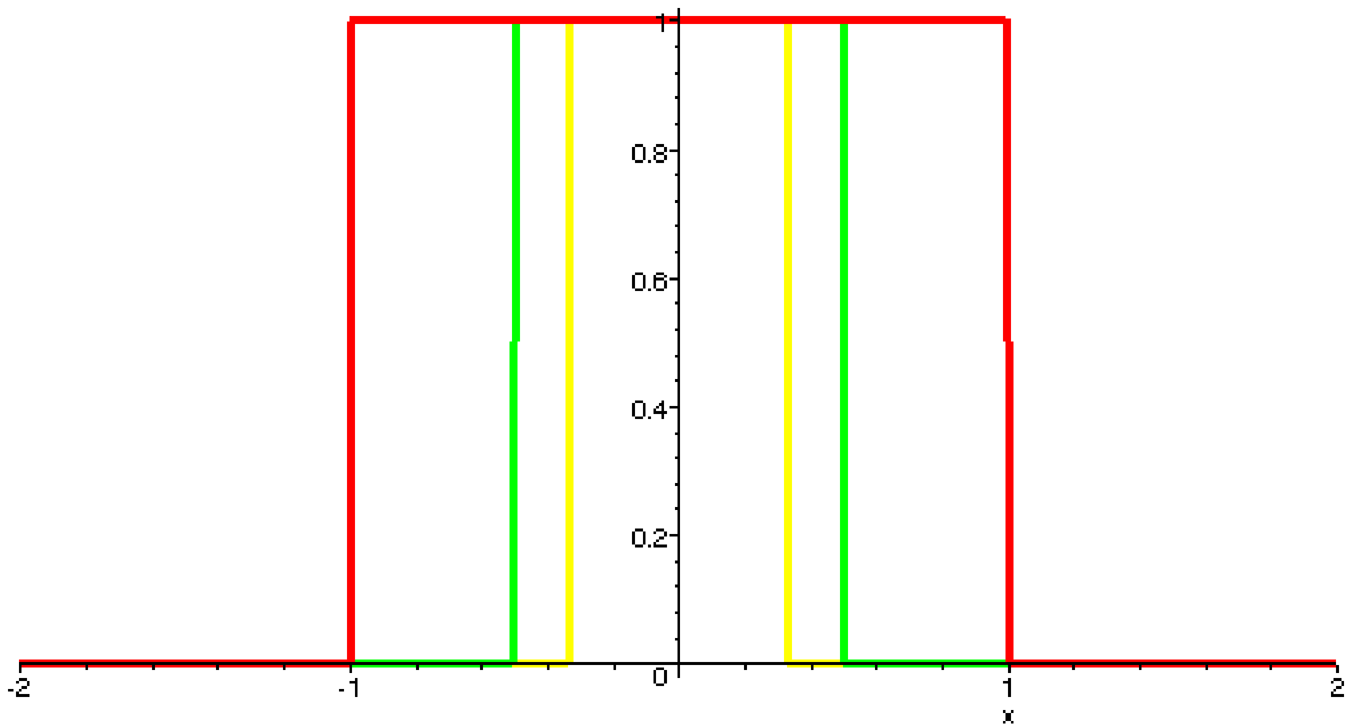
Can limits can be interchanged?

Recall that there could be a problem with interchanging limits.

Example

$$S[n](x) = \text{Heaviside}(x-1/n) - \text{Heaviside}(x+1/n).$$

Imagine the graph of the first three terms of this sequence.



Graph of Heaviside(x-1/n) - Heaviside(x+1/n).

Choose an n.

Compute the limit as $x \to 0$ for $S[n](x)$. This limit is one.

Thus, the limit as $n \to \infty$ of the limit as $x \to 0$ of $S[n](x)$ is one.

Now we do it the other way.
Pick an x different from zero. Consider the limit as $n \to \infty$. For example, take n so large that $1/n < |x|$.
It follows that the limit as $x \to 0$ of the limit as $n \to \infty$ infinity is zero.
This is not the same answer as when the limits were taken in the reverse order.

We see that the limits cannot be interchanged in this example. It matters whether we take the limit as $x \to 0$ first, or as $n \to \infty$ first.

THEOREM. If f(x) is periodic, continuous, and sectionally smooth, then the differentiated Fourier series of f(x) converges to f '(x) at every point s where f ''(x) exists.

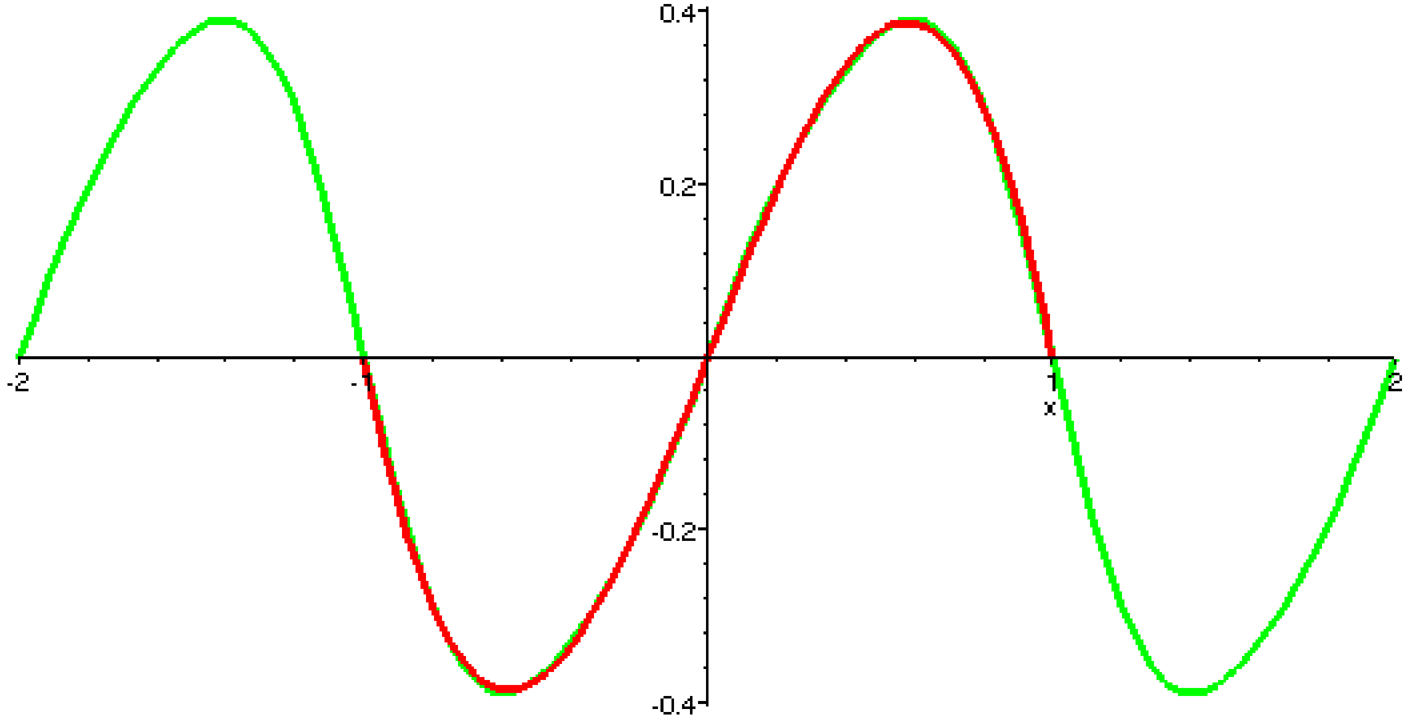
Example: Consider the function

f(x) = $\mathbf{x - x^3}$ on the interval [-1, 1].

No cosine terms in its Fourier series.

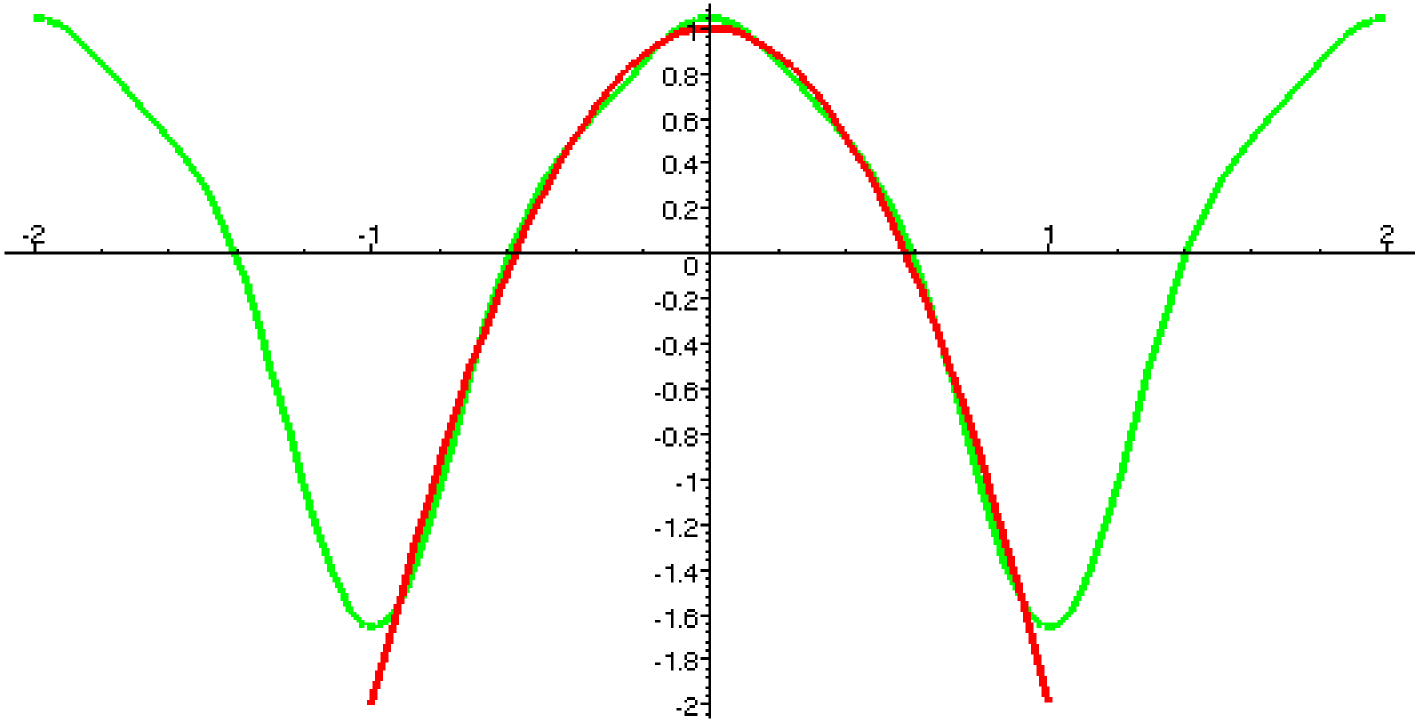
Coefficients: $$\frac{12\ (-1)^{p+1}}{p^3 \pi^3}$$

Plot three terms and compare the graphs.



Graph of $\mathbf{x - x^3}$ and the Fourier series.

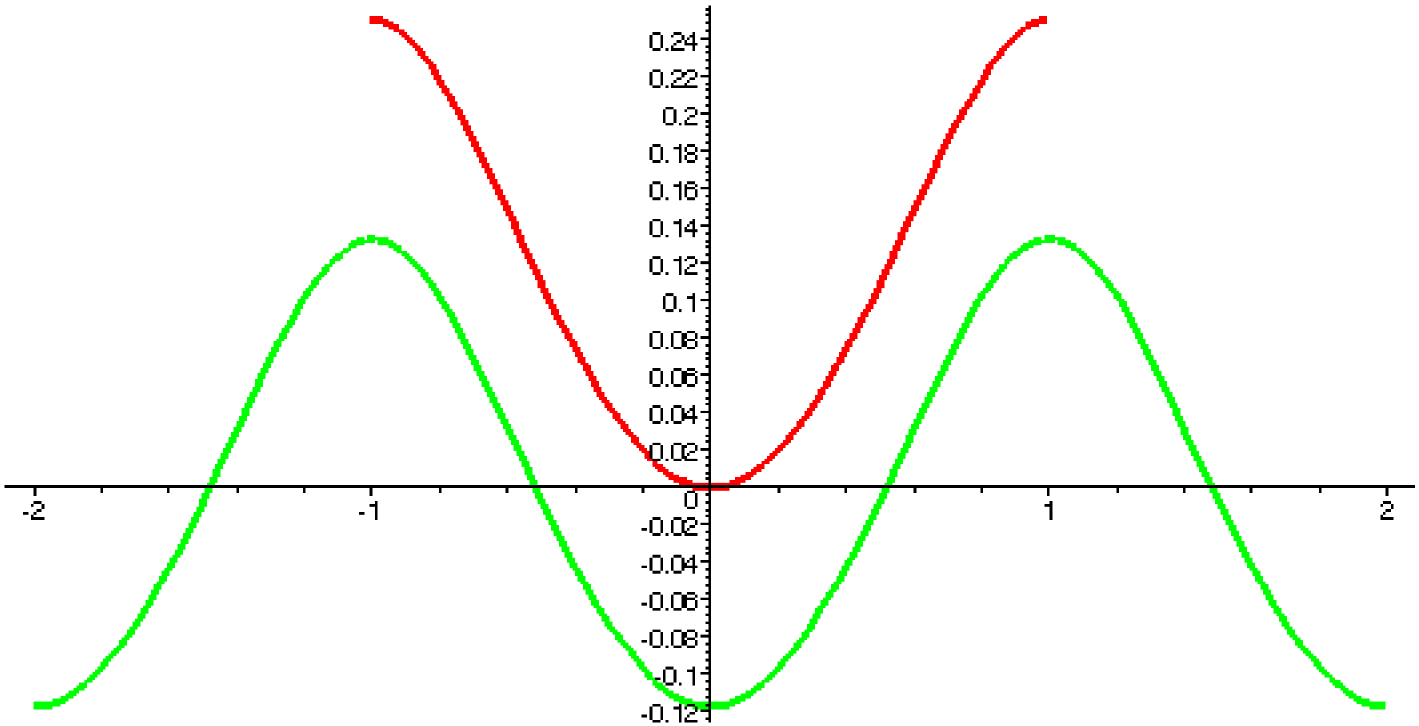
What happens if we differentiate?



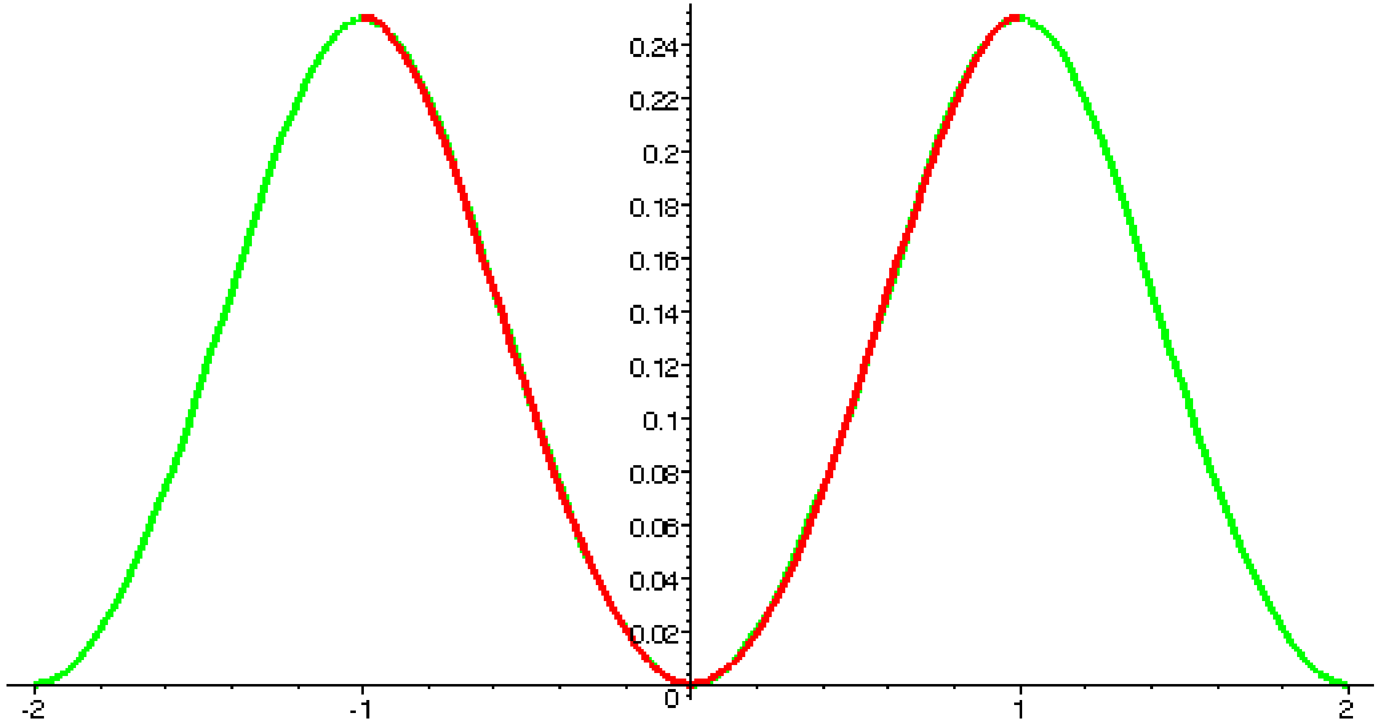
What about integration?

THEOREM: if f(x) is periodic and sectionally continuous, then the Fourier series for f(x) may be integrated term-by-term.

Example: We take the antiderivatives.



Not To Worry!



There was no real problem.

Assignment: See the Maple Worksheet

In this tenth module, we have discussed

1. differentiation of a series term-by-term
2. integration of a series term-by-term.