Module 11: Generation of Orthogonal Families

Orthogonal families are useful.

Generating an orthogonal family with the Gramm-Schmidt process.

We introduce another method:
compute eigenvectors for self-adjoint, linear transformations.

Definition: A function A on a vector space is linear if

$$A(\lambda x + y) = \lambda A(x) + A(y);$$

for all numbers λ and all vectors x and y.

Examples.

1. Let A be a square matrix defined on a finite dimensional vector space.
2. Let the space be $C''([0,\infty))$ and $A(f) = f''$ for any f in the space.

Definition: A linear function A is self-adjoint on the space { E, < *, * > } if , for all x and y in the domain of the operator

$$\langle Ax, y \rangle = \langle x, Ay \rangle$$

3. Example: Matrices in the form

$$\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}.$$

4. Example: Twice differential operator on a subset of C ''([0, 1]) with the usual dot product.

Domain: a subset of C ''([0, 1]) consists of functions f such that f(0) = f(1) = 0.

A is A(f) = f ''.

To see why: remember integration-by-parts:

$$\int_a^b u' \, v \, dx = u(b)\, v(b) - u(a)\, v(a) - \int_a^b u \, v' \, dx$$

$$\int_a^b f''(x)\, g(x)\, dx =$$

$$f'(b)\, g(b) - f'(a)\, g(a) - \int_a^b f'(x)\, g'(x)\, dx =$$

$$- \int_a^b f'(x)\, g'(x)\, dx =$$

$$- f(b)\, g'(b) + f(a)\, g'(a) + \int_a^b f(x)\, g''(x)\, dx =$$

$$\int_a^b f(x)\, g''(x)\, dx$$

We have established that this A --
two derivatives and boundary conditions --
is self adjoint. Other boundary conditions could achieve this same condition.

Definition: The number λ is an eigenvalue and the vector v is an eigenvector of the linear transformation A if

$$A(v) = \lambda v.$$

Example 5.

The numbers -1 and -2 are eigenvalues corresponding to the eigenvectors

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

for

$$\begin{pmatrix} -3/2 & 1/2 \\ 1/2 & -3/2 \end{pmatrix}.$$

Example 6.

The infinite sequence $-n^2\Pi^2$ are all eigenvalues for

$$A(f) = f'',$$

with boundary conditions $f(0) = 0$ and $f(1) = 0$.

Associated eigenvectors (eigenfunctions) are

$$\sin(n\pi x).$$

Derivation later!

We do not forget the purpose of this development.

Fact 1: Eigenvalues corresponding to self-adjoint transformations are real numbers - not complex numbers.

Fact 2: Eigenvectors for self adjoint transformations corresponding to different eigenvalues are orthogonal.

Revisit Examples 5 and 6

Definition of a regular Sturm-Liouville problem:

Suppose that s, s ', and q are continuous and s(x) is not zero. With boundary conditions,

$$L(f) = (s f ')' - q f$$

is self adjoint in C ''([a,b]).

Possible boundary conditions

(1) $f(0) = f(1) = 0$, or

(2) $f'(0) = f'(1) = 0$, or

(3) $f(0) = \alpha f'(0)$ and $f(1) = -\alpha f'(1)$, or

(4) if $s(0) = s(1)$, then $f(0) = f(1)$ and $f'(0) = f'(1)$.

Take $\lambda = -n^2\pi^2$

and

$y = \sin(n\pi x)$

or

$y = \cos(n\pi x)$

These satisfy

$y'' = \lambda y$

$y(-1) = y(1)$ and $y'(-1) = y'(1)$.

Assignment: See the Maple worksheet.

In this Module 11, we

1. defined eigenvalue and eigenvector,
2. defined a self adjoint transformation,
3. obtained orthogonal families, and
4. took note of Sturm-Liouville problems.