Module 12: Review of Elementary Differential Equations

General Solutions for Linear, Constant Coefficient Differential Equations.

Recall the techniques for

$$y'' + 3y' + 2y = 0.$$

Guess the solution is of the form $y = \exp(r x)$.

Consider the nullspace of the linear operator

$$L(y) = y'' + 3y' + 2y.$$

The nullspace is two dimensional and any solution of the equation -- any element of the nullspace -- can be written as a linear combination of these two.

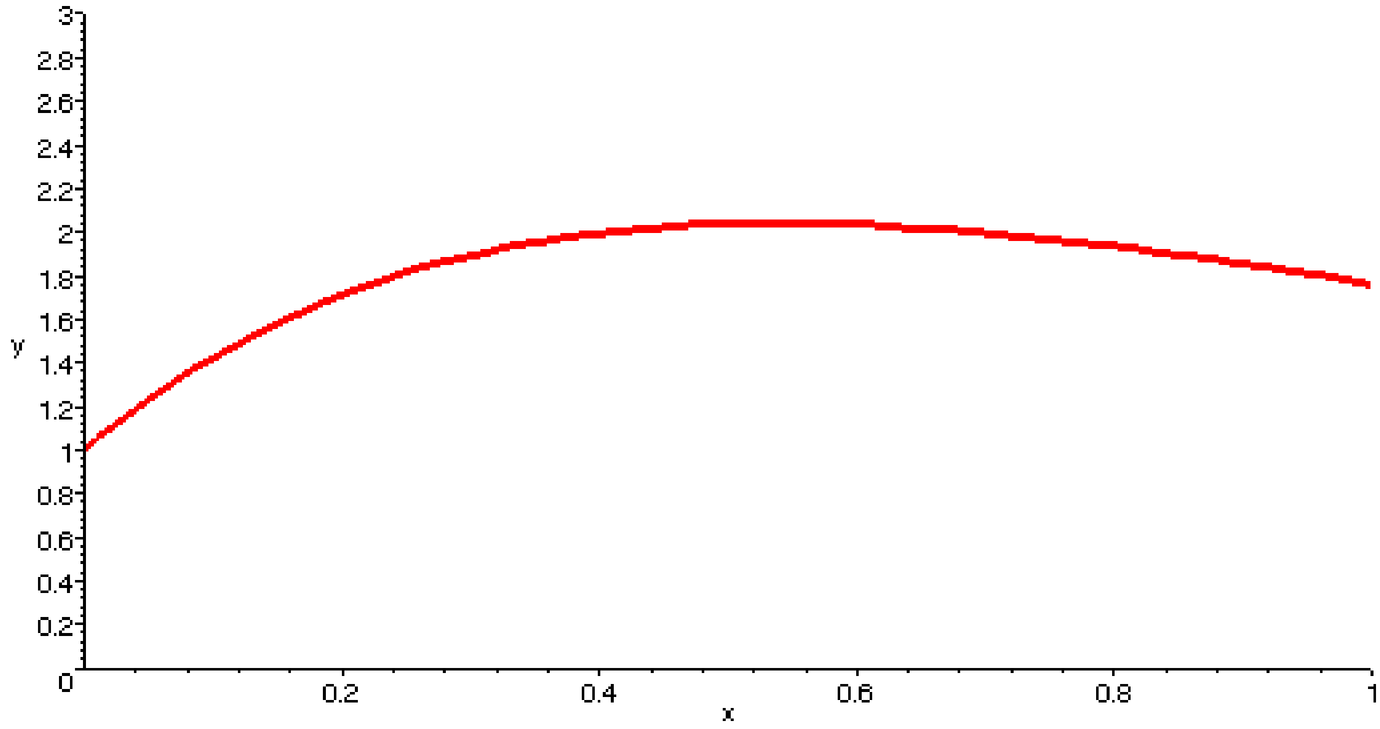
Linear, Constant Coefficient, Initial Value Problems

If we want to know a particular solution, say the solution of an initial value problem, we must choose the constants so that the initial values are attained.

$y'' + 3y' + 2y = 0, \; y(0) = 1, \; y'(0) = 5.$

Choose A and B in $A \exp(-x) + B \exp(-2x)$.

A graph of this particular solution.



Linear, Constant Coefficient, Boundary Value Problems.

There are second order, constant coefficient differential equations for which there is exactly one solution, for which there is no solution, and for which there is an infinity of solutions for BOUNDARY VALUE PROBLEMS

A solution for

$$y'' + 3y' + 2y = 0,\ y(0) = 1,\ y(1) = 1.$$

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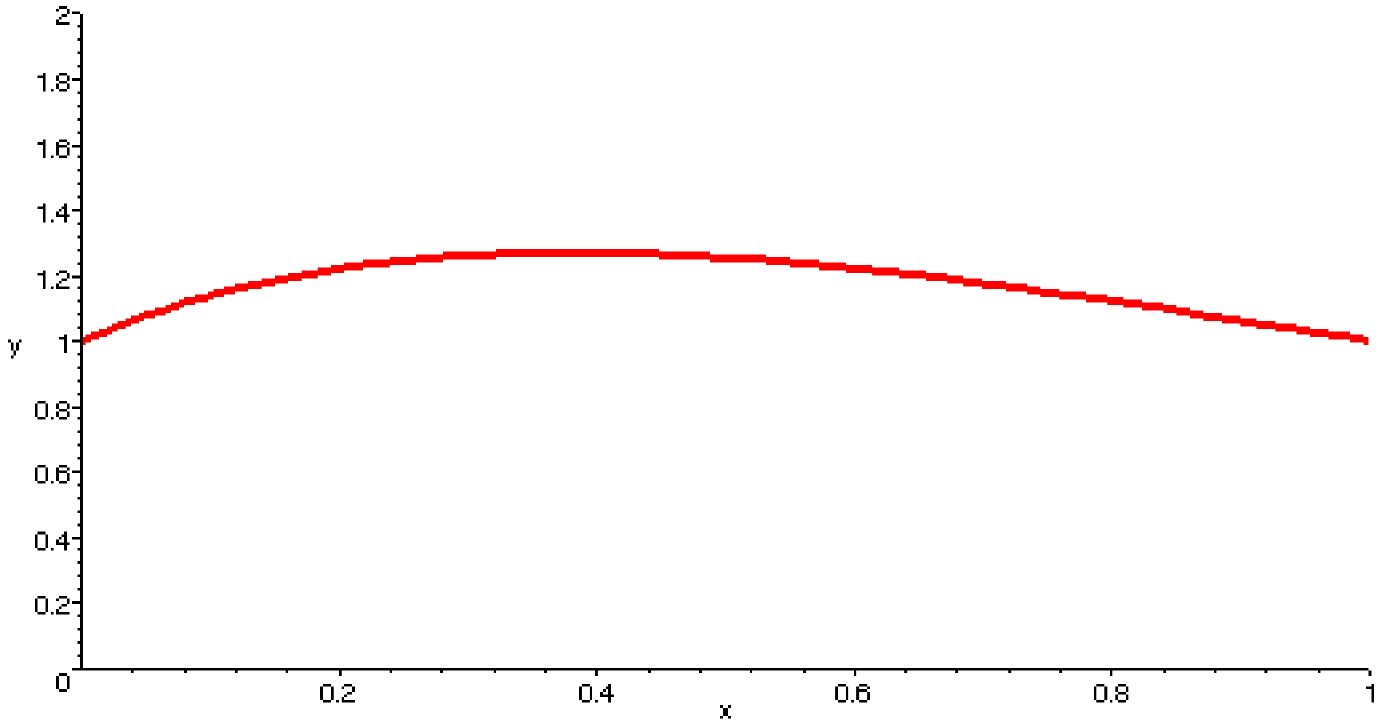
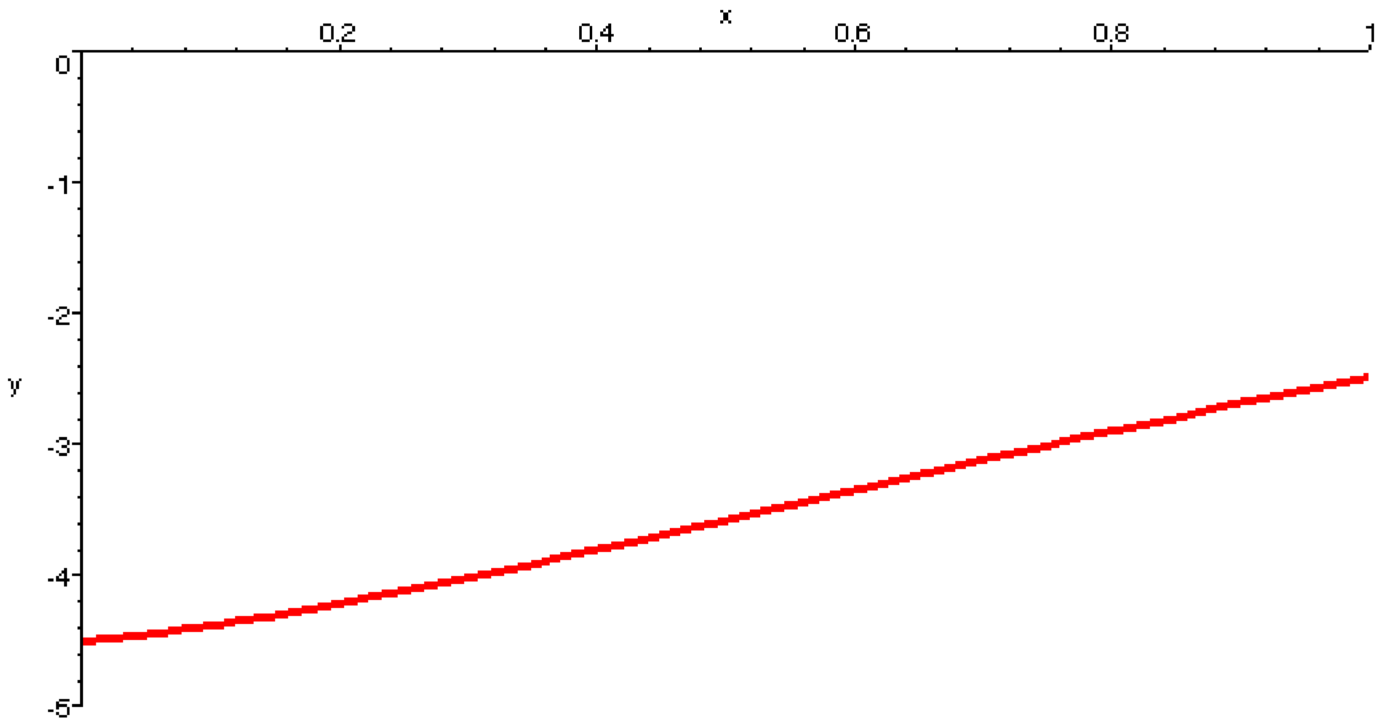


$$y'' + 3y' + 2y = 0,\ y'(0) = 1,\ y'(1) = 2.$$



We can make second order, constant coefficient, differential equations for which there is no solution for the boundary value problem:

$$y'' + 3y' + 0y = 0, \quad y'(0) = a, \quad y'(1) = b.$$

General solution: $\alpha + \beta \exp(-3x)$.

There are second order, constant coefficient, differential equations for which there are an infinity of solution.

$$y'' = 0, \quad y'(0) = 0, \quad y'(1) = 0.$$

Linear Systems of Differential Equations

Review your experience. Here is an example

X'(t) = -2 X(t) - 11 Y(t),

Y'(t) = 11 X(t) - 2 Y(t),

X(0) = 1, Y(0) = 2.

Or, $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -2 & -11 \\ 11 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$

Non-homogeneous Systems.

The right hand side is not zero. Example: Newton's Law of Cooling.

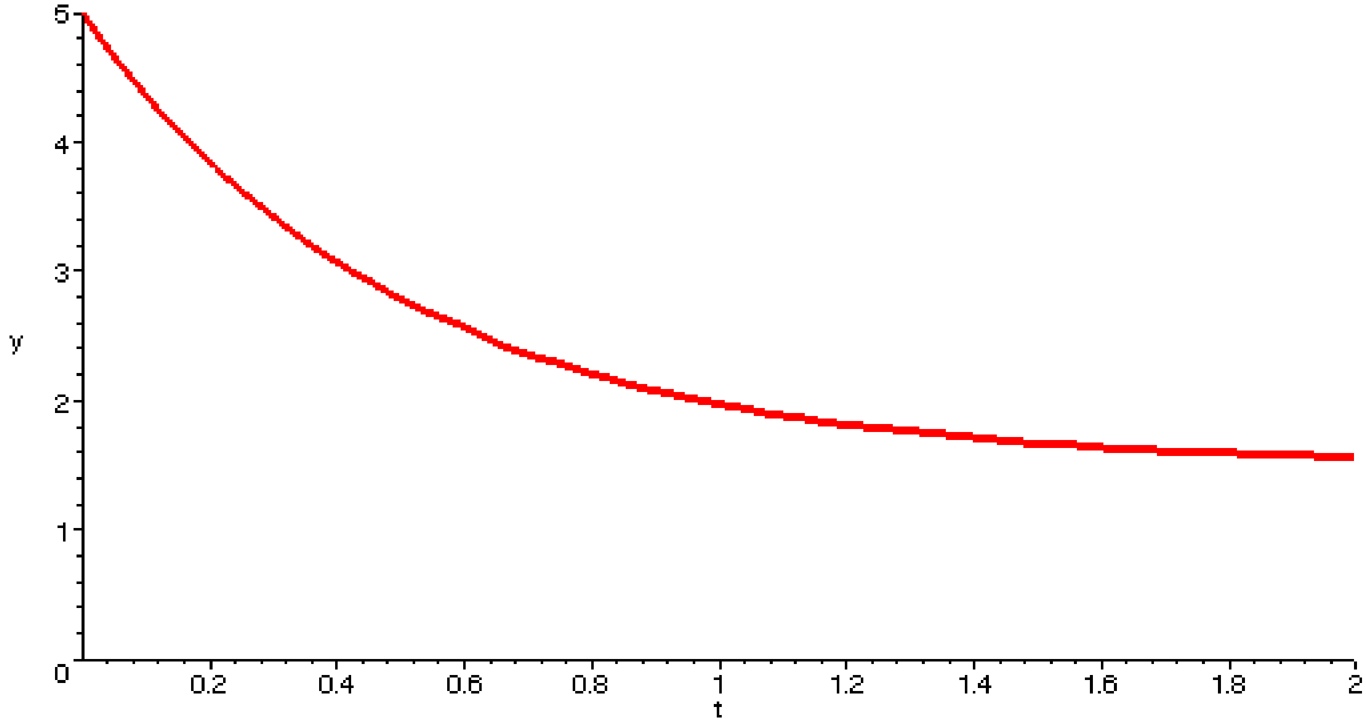
$T' = K(A - T)$, $T(0) = C$

K is positive. Rewrite as

$$T' + KT = KA, \text{ with } T(0) = C.$$

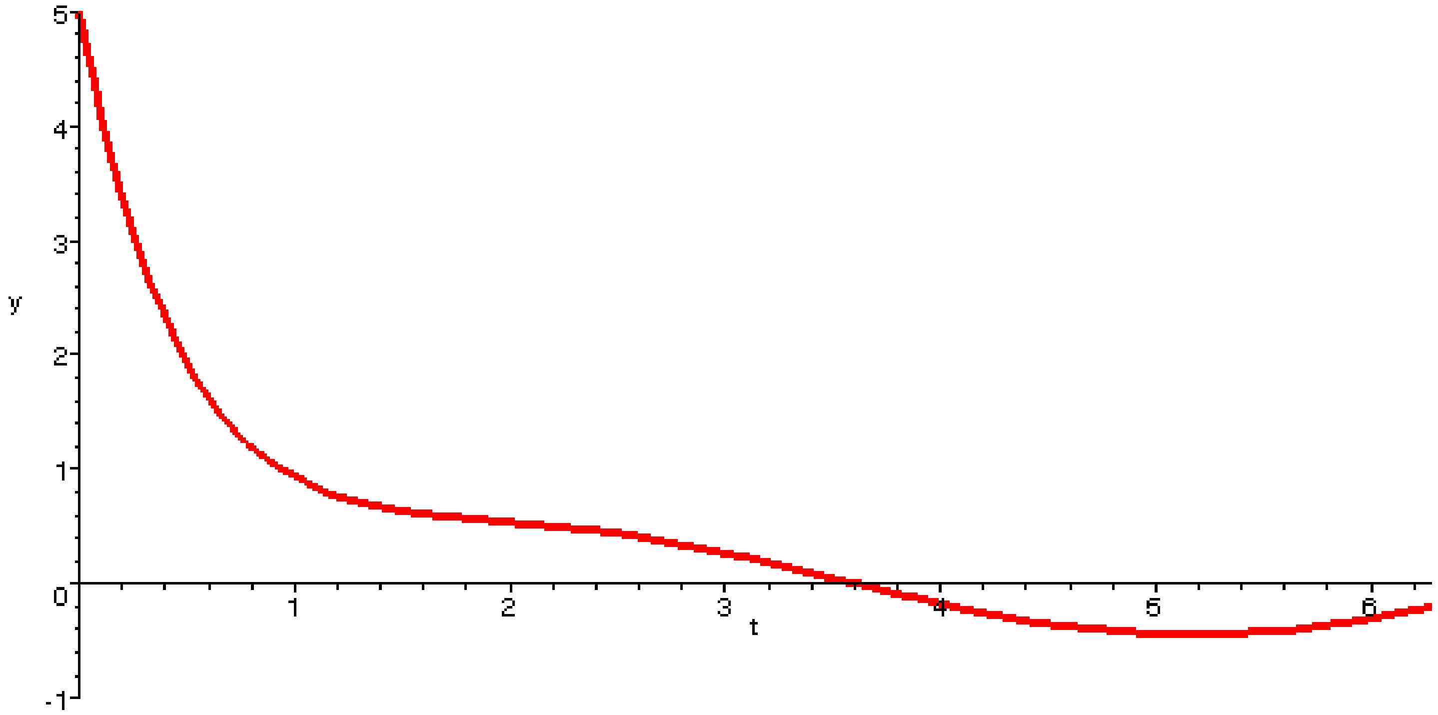
Example: $T' + 2T = 3$, $T(0) = 5$.

Solution for T ' + 2 T = 3, T(0) = 5.



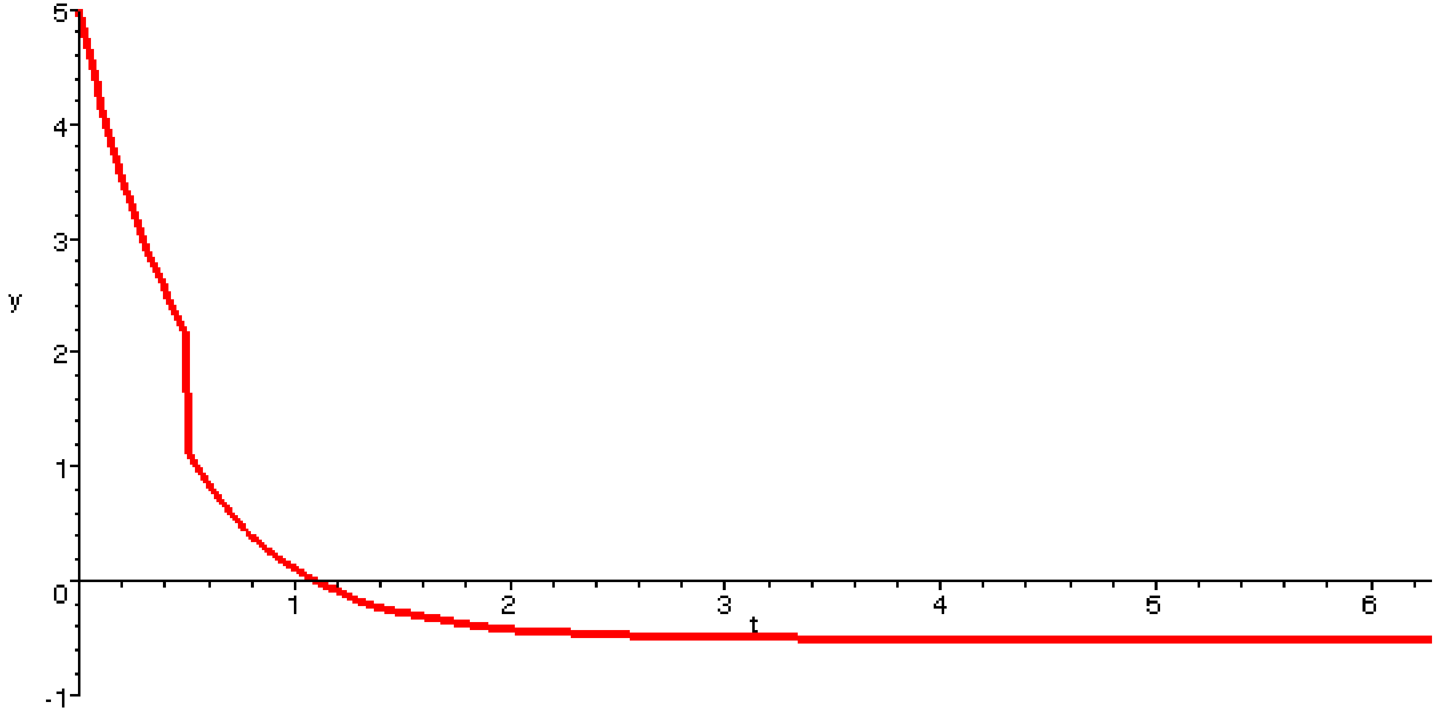
The right side may, itself, be a function of t.

$$T' + 2T = \sin(t),\ T(0) = 5.$$



The right side may be not differentiable.

$$T' + 2T = \mathrm{signum}(1/2 - t),\ T(0) = 5.$$



Numerical solutions for Differential Equations

The previous example. Numerical solutions can be used for systems of equations, too.

$x'(t) = \sin(t) - 2\, x(t) - 3\, y(t),$
$y'(t) = 3\, x(t) - 2\, y(t),$
$x(0) = 1,\ y(0) = 0.$

This example might have a biological origin.

Assignment: See Maple

In this 12th Module, we have reviewed ordinary differential equations:

1. initial value problems,
2. boundary value problems,
3. systems of equations, and
4. numerical solutions.