Module 13: Review of Elementary Differential Equations II

Question 1. Suppose $\lambda > 0$. Which of these is a pair of linearly independent solutions for

$$Y'' - \lambda^2 Y = 0 \text{ on } [0, \pi]?$$

a. $\exp(\lambda x)$ and $\exp(-\lambda x)$,
b. $\sin(\lambda x)$ and $\cos(\lambda x)$,
c. $\sinh(\lambda x)$ and $\cosh(\lambda x)$,
d. $\sinh(\lambda x)$ and $\sinh(\lambda (\pi - x))$

Question 2. Suppose $\lambda > 0$. Which of these is a pair of linearly independent solutions for

$$Y'' + \lambda^2 Y = 0 \text{ on } [0, \pi]?$$

e. $\exp(\lambda x)$ and $\exp(-\lambda x)$,

f. $\sin(\lambda x)$ and $\cos(\lambda x)$,

g. $\sinh(\lambda x)$ and $\cosh(\lambda x)$,

d. $\sinh(\lambda x)$ and $\sinh(\lambda (\pi - x))$

Question 3. Suppose $\lambda > 0$. Which of these is a bounded solution for

$$Y'' - \lambda^2 Y = 0 \text{ on } [0, \infty)?$$

a. $\exp(\lambda x)$ b. $\exp(-\lambda x)$

c. $\sinh(\lambda x)$ d. $\cosh(\lambda x)$

There are two issues here: which is a solution and which is bounded on the specified interval.

Question 4. Which of these is a bounded solution on the interval [0, 5] for the differential equation

$$r^2 R''(r) + r R'(r) - 9 R(r) = 0?$$

a. exp(3 r) b. r^3; c. sin(3 r)

d. exp(–3 r) e. $1/r^3$; f. cosh(3 r)

Question 5. If $u(x, y) =$

$$\sum_p a_p \sin(px) \sinh(py) + \sum_p b_p \sin(px) \sinh(p(\pi - y))$$

and

$$u(x,0) = 0, \quad u(x, \pi) = \sin(2x)$$

what are the a_p's and b_p's ?

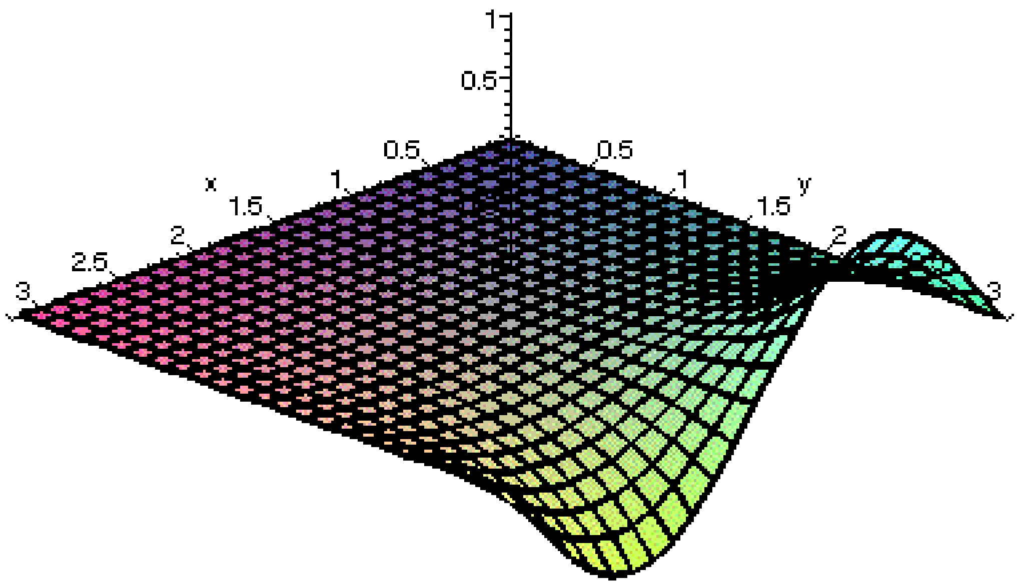
We have two pieces of information.

$$\sum_p a_p \sin(px)\sinh(py) + \sum_p b_p \sin(px)\sinh(p(\pi - y))$$

$$u(x,0) = 0, \quad u(x,\pi) = \sin(2x)$$

The first of these implies that all the b_p's = 0, and the second implies that all the a_p's =0, except $a_2 = 1/\sinh(2\pi)$.

Graph of sin(2 x) sinh(2 y) / sinh(2 π)



Question 6. If $u(r, \theta) =$

$$\sum_p a_p \sin(p\,\theta)\, r^p + \sum_p b_p \cos(p\,\theta)\, r^p$$

and

$$u(1, \theta) = 1 + 3 \cos(3\,\theta) + 5 \sin(2\,\theta)$$

then what is $u(r, \theta)$, $u(0,0)$, and $u(1/2, \pi/4)$?

$$\sum_p a_p \sin(p\,\theta)\, r^p + \sum_p b_p \cos(p\,\theta)\, r^p$$

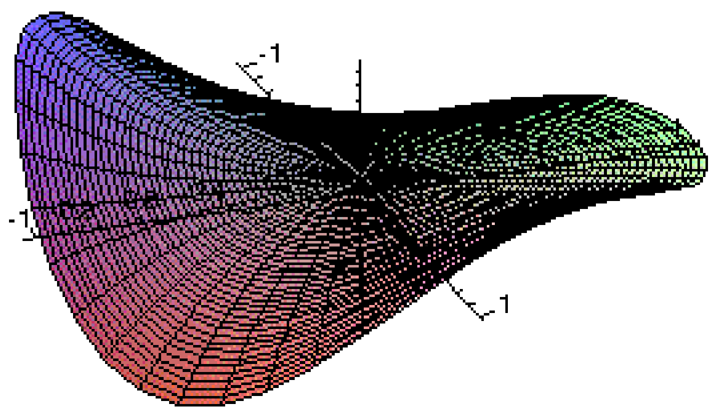
$$u(1, \theta) = 1 + 3\cos(3\,\theta) + 5\sin(2\,\theta)$$

This implies that all a_p's $=0$ and b_p's $= 0$

except $b_0 = 1$, $b_3 = 3$, and $a_2 = 5$.

$u(r, \theta) = 1 + 3\,r^3 \cos(3\,\theta) + 5\,r^2 \sin(2\,\theta)$

$u(0,0) = 1$ and $u(1/2, \pi/4) = 1 - 3/8\,\sqrt{2} + 5/4.$



Question 7. What are all the eigenvalues of the self-adjoint, Sturm-Liouville Problem

$$y'' = \mu y, \text{ with } y(0) = y(1) = 0?$$

We break the problem into two cases.

First, suppose that $\mu > 0$.

Take $\mu = \lambda^2$. Thus, we seek numbers λ such that

$$y'' = \lambda^2 y \text{ with } y(0) = y(1) = 0.$$

$$y'' = \lambda^2 y \text{ with } y(0) = y(1) = 0.$$

has general solutions of the form

$$y(x) = A \exp(\lambda x) + B \exp(-\lambda x).$$

$y(0) = 0 \qquad 0 = A + B.$

$y(1) = 0 \qquad 0 = A \exp(\lambda) + B \exp(-\lambda).$

Thus $A = 0 = B$.

$y'' = \mu y$, with $y(0) = y(1) = 0$?

Second case: $\mu < 0$. Take $\mu = -\lambda^2$.

$y'' = -\lambda^2 y$ with $y(0) = y(1) = 0$.

General solution is

$$y(x) = A \sin(\lambda x) + B \cos(\lambda x).$$

$y(0) = 0$ $0 = B$.

$y(1) = 0$ $0 = A \sin(\lambda)$.

$0 = \sin(\lambda)$ so that $\lambda = n\pi$ and $\mu = -n^2 \pi^2$.

Eigenvalues are = – $n^2 \pi^2$ and

eigenfunctions are sin(n π x).

Surprised?

Assignment: See Maple Worksheet

In this Module 13 we have

1. Examined bounded solutions for Sturm-Liouville Problems,
2. Found eigenvalues for a Sturm-Liouville Problem, and
3. Identified coefficients of trigonometric series.