Module 14: First Order, Non-homogeneous, Initial Value Problems

Problem 1. Graph the solution for the differential equation $y' = -2y + 3$, $y(0) = 5$.

Step 1: Find a particular (independent of time) solution

$$0 = -2y + 3, \text{ so that } y = 3/2.$$

Step 2: Find the general solution for the homogeneous equation

$$y' = -2y, \quad \text{so that } y(t) = \exp(-2t)\, C.$$

Step 3: Add these two to get the general solution for the non-homogeneous equation, so that

$$y(t) = \exp(-2t)\, C + 3/2.$$

General Solution: $y(t) = \exp(-2\,t)\,C + 3/2$.

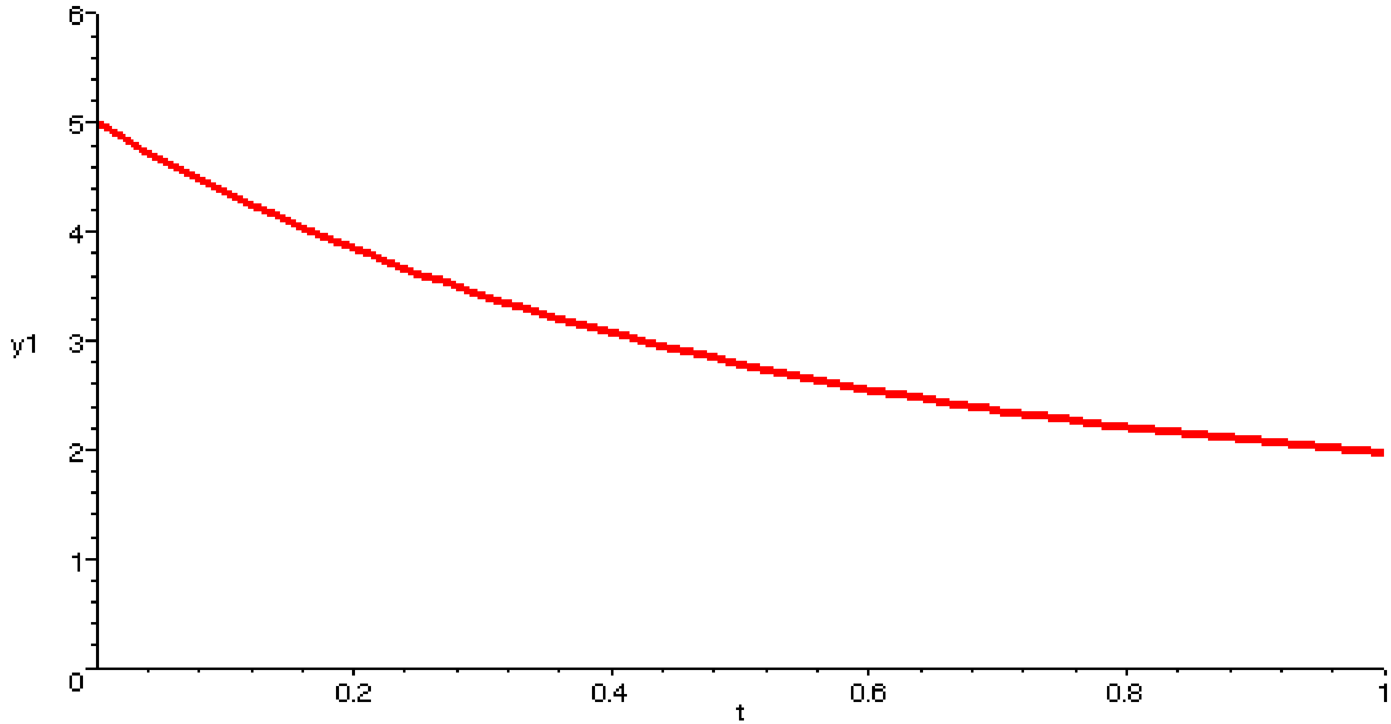
Step 4: Find the solution for the non-homogeneous equation which satisfies the initial condition, so that

$$5 = y(0) = C + 3/2 \text{ and } C = 7/2.$$

Check: $y' = -2\,y + 3,\ y(0) = 5$

$$y(x) = \exp(-2\,t)\,7/2 + 3/2$$

Graph of solution for y' = - 2 y + 3, y(0) = 5.



Problem 2. Suppose that A is a matrix which has an inverse and that v is a vector. Graph the solution for the differential equation

$$Z' = A Z + v, \quad Z(0) = [2,3].$$

Step 1: Find a particular (independent of time) solution

$$0 = A Z + v, \text{ so that } Z = - A^{-1} v.$$

Step 2: Find the general solution for the homogeneous equation

$$Z' = A\,y, \quad \text{so that } Z(t) = \exp(A\,t)\,C.$$

Step 3: Add these two to get the general solution for the non-homogeneous equation, so that

$$Z(t) = \exp(A\,t)\,C - A^{-1}\,v\,.$$

General Solution: $Z(t) = \exp(A\, t)\, C - A^{-1} v$.

Step 4: Find the solution for the non-homogeneous equation which satisfies the initial condition,

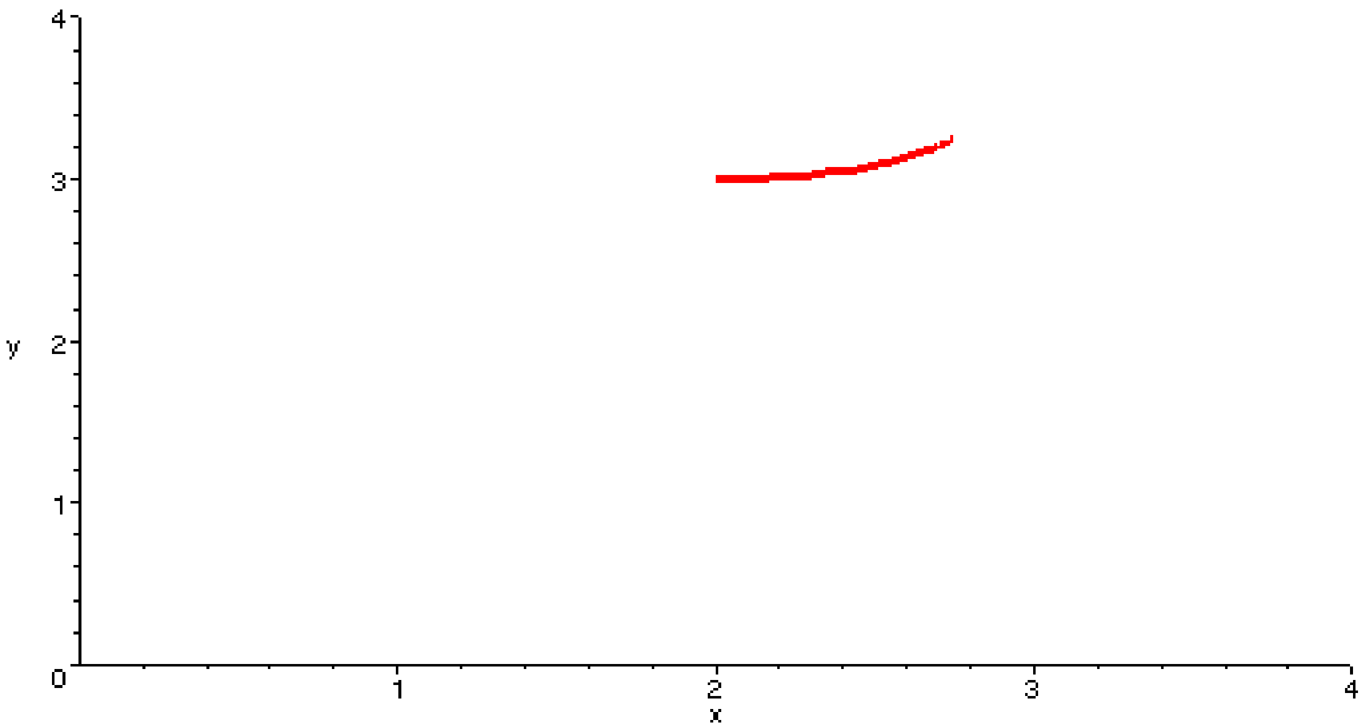
so that

$$[2, 3] = Z(0) = C - A^{-1} v$$

and $C = [2, 3] + A^{-1} v$.

Take $A = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}$ and $v = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$.

Initial value: $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$



Problem 3: Graph the solution for the partial differential equation

$$du/dt = d^2u/dx^2 \ , \quad u(t, 0) = 3, \ u(t, 2) = 5,$$

with u(0, x) = f(x).

(Non-homogeneous boundary conditions. Homogeneous boundary conditions would be

u(t, 0) = 0, u(t, 2) = 0.)

Step 1: Find a particular (independent of time) solution

$$0 = d^2u/dx^2, \quad \text{with } u(0) = 3 \text{ and } u(2) = 5.$$

ODE: $0 = u''$, with boundary conditions.

The solution for this equation is

$$u(x) = x + 3.$$

Step 2: Find the general solution for the homogeneous equation

$$du/dt = d^2u/dx^2 \;, \quad u(t, 0) = 0, \; u(t, 2) = 0,$$

We will see later that the general solution for this equation is

$$u(t, x) = \sum C_n \exp(-n^2 \Pi^2 t/4) \sin(n\Pi x/2).$$

Check this?

Step 3: Add these two to get the general solution for the non-homogeneous equation, so that

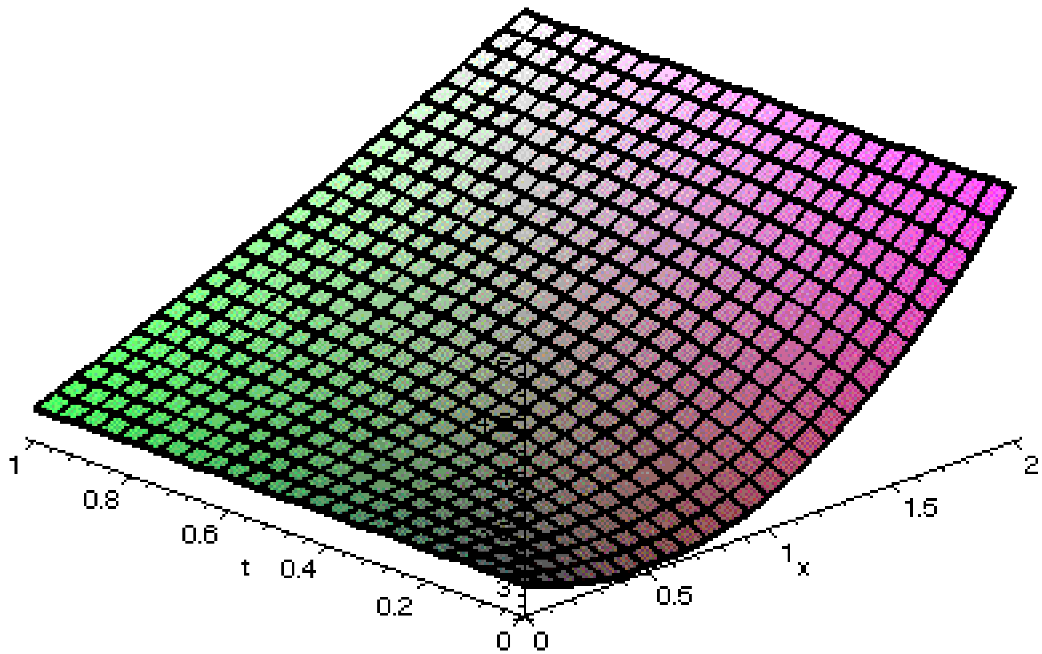
$$u(t, x) = x + 3 + \sum C_n \exp(-n^2 \Pi^2 t/4)\sin(n\Pi x/2)$$

Step 4: Find the solution for the non-homogeneous equation with initial :

$$f(x) = x + 3 + \sum C_n \sin(n\Pi x/2) \text{ for } 0 < x < 2.$$

This looks like a job for Fourier Series.

Graph of u with $f(x) = x^2 - x + 3$



Assignment: See Maple Worksheet

In this Module 14, we have

1. Solved a non-homogeneous differential equation.
2. Solved a non-homogeneous differential equation system.
3. Solved a partial differential equation with non homogeneous boundary conditions.