Module 15: The Critical Notion: Complete Orthonormal Sequences

Definition: The collection of vectors

$$\{ \Theta_1, \Theta_2, \Theta_3, \ldots \}$$

is called orthogonal if

$$< \Theta_i, \Theta_j > = 0, \text{ whenever } i \neq j.$$

The family is orthonormal if each $|\Theta_i| = 1$.

Here is the important inequality that we derived:

$$\left\| f - \sum_p a_p \Phi_p \right\|^2 = \|f\|^2 + \sum_p \left| \left(\langle f, \Phi_p \rangle - a_p \right) \right|^2 - \sum_p \left\| \langle f, \Phi_p \rangle \right\|^2$$

Several important facts followed from this inequality.

Fact 1. Suppose that n is a positive integer, that

$$\{ \Phi_1, \Phi_2, \Phi_3, \ldots, \Phi_n \}$$

is an orthonormal sequence, and that

$$S_n = \text{span of the } \{ \Phi_1, \Phi_2, \Phi_3, \ldots, \Phi_n \}.$$

If f is in the space, then the closest element in S_n to f is given by the Fourier expansion:

$$\sum_{p=1}^{n} \langle f, \Phi_p \rangle \Phi_p$$

We argue this:

$$\left\| f - \sum_p a_p \Phi_p \right\|^2 = \|f\|^2 + \sum_p \left| \left(\langle f, \Phi_p \rangle - a_p \right) \right|^2 - \sum_p \left\| \langle f, \Phi_p \rangle \right\|^2$$

Fact 2. If the Φ_p 's form an infinite sequence, then the infinite series

$$\sum_p \left|\langle f, \Phi_p \rangle\right|^2$$

converges.

We argue this:

$$\left\| f - \sum_p a_p \Phi_p \right\|^2 = \|f\|^2 + \sum_p \left|\langle f, \Phi_p \rangle - a_p\right|^2 - \sum_p \left\|\langle f, \Phi_p \rangle\right\|^2$$

Fact 3. If the Φ_p 's form an infinite sequence, then the infinite series

$$\sum_{p=1}^{\infty} \langle f, \Phi_p \rangle \Phi_p$$

in the vector space converges.

We argue this: if n > m then

$$\left\| \sum_{p=m}^{n} \langle f, \Phi_p \rangle \Phi_p \right\|^2 = \sum_{p=m}^{n} \left| \langle f, \Phi_p \rangle \right|^2$$

and this latter converges.

Definition: An orthogonal sequence is <u>complete</u> if the only vector in the space that is orthogonal to every element in the sequence is the zero vector.

Fact 4. Suppose $\{ \Theta_1, \Theta_2, \Theta_3, \ldots \}$ is a complete orthonormal sequence. Then

$$\sum_{p=1}^{\infty} \langle f, \Phi_p \rangle \Phi_p$$

converges to f in norm.

We argue this.

We argue that

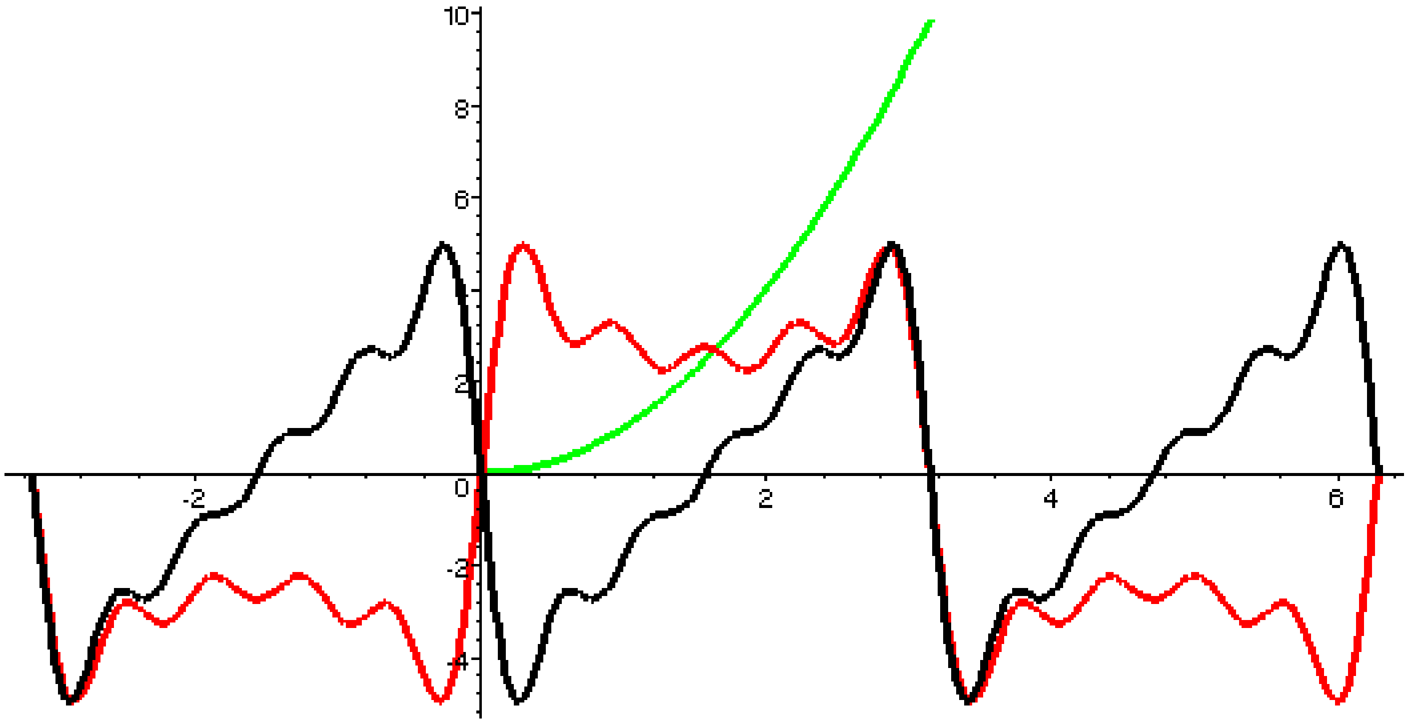
$$\sum_{p=1}^{\infty} \langle f, \Phi_p \rangle \Phi_p = f$$

by showing that the element g defined as

$$f - \sum_{p=1}^{\infty} \langle f, \Phi_p \rangle \Phi_p$$

is orthogonal to each Φ_i

The projection of f onto the span of sin(nx)'s and onto the span of sin(mx)'s where n is odd and m is even.



Assignment: See Maple Worksheets.

In this Module 15, we have provided an argument in support of the proposition that the Fourier Series for a function f, using a complete, orthonormal family, converges in norm to f.