Module 16: The Simple Heat Equation

We consider the simple heat equation:

$$du/dt = d^2u/dx^2, \quad u(t, 0)=0, \quad u(t, 1) = 0$$

$$u(0, x) = f(x).$$

There is a physical interpretation.

The method of solving this equation is called:
Separation of Variables.

$$u(t,x) = X(x)\,T(t).$$

$du/dt = d^2u/dx^2$ leads to

$$X\,T' = X''\,T,$$

or

$$T'/T = X''/X.$$

$T' / T = X'' / X$ T' / T and X'' / X are constant.

$U(t, 0) = X(0)\,T(t) = 0$ $X(0) = 0$

$U(t, 1) = X(1)\,T(t) = 0$ $X(1) = 0$

$$X'' = \mu X, \text{ with } X(0) = 0 = X(1).$$

See Lecture 13, Question 7. There:

μ's are $-n^2 \pi^2$ and

x's are $\sin(n \pi x)$.

What about the possible T solutions?
The quotient T '/ T is the same quotient,
so that

$$T' = -n^2 \pi^2 T.$$

Solutions for this equation are

$$T(t) = \exp(-n^2 \pi^2 t).$$

Consequently, for each integer n, we have a solution.

For each integer n, we have a solution.

$$u(t, x) = \exp(-n^2 \pi^2 t) \sin(n \pi x) .$$

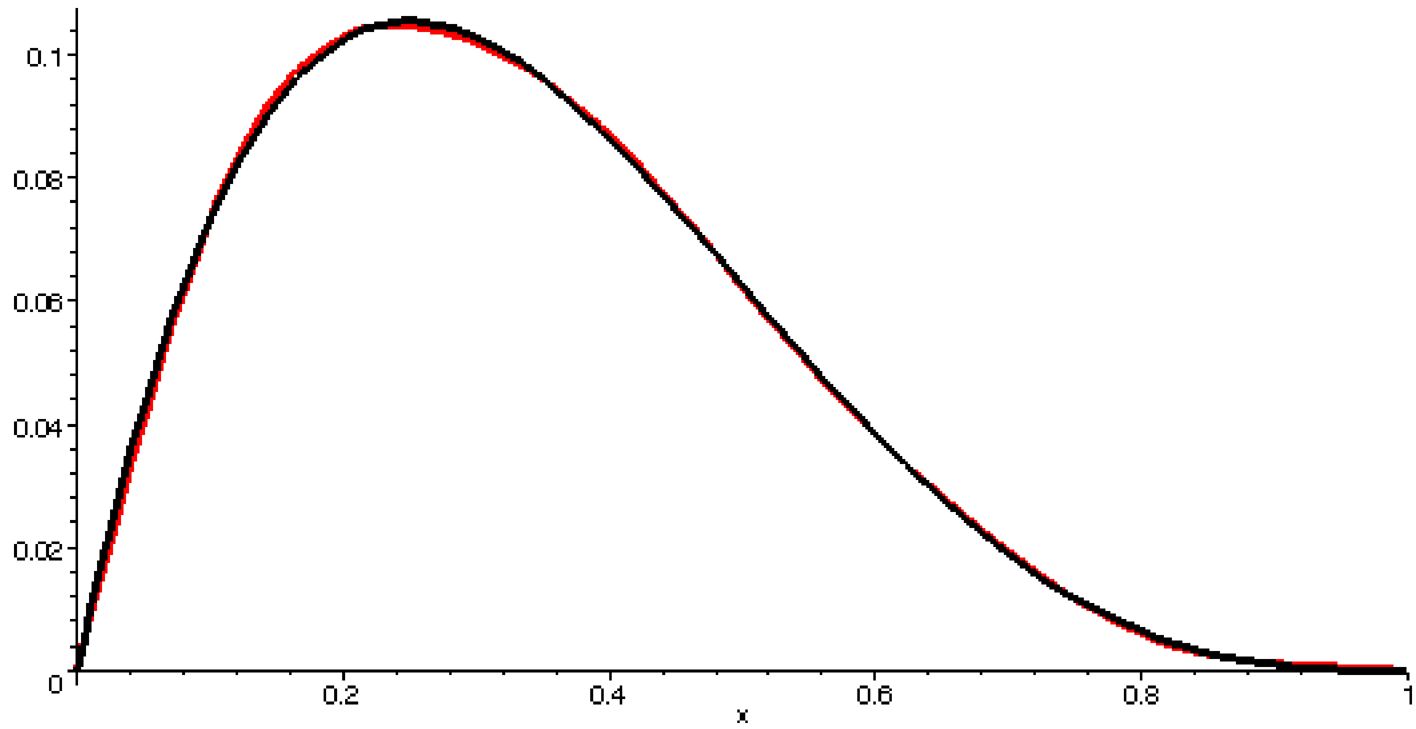
Conclusion:

$$u(t, x) = \sum_n c_n \exp(-n^2 \Pi^2 t) \sin(n \Pi x)$$

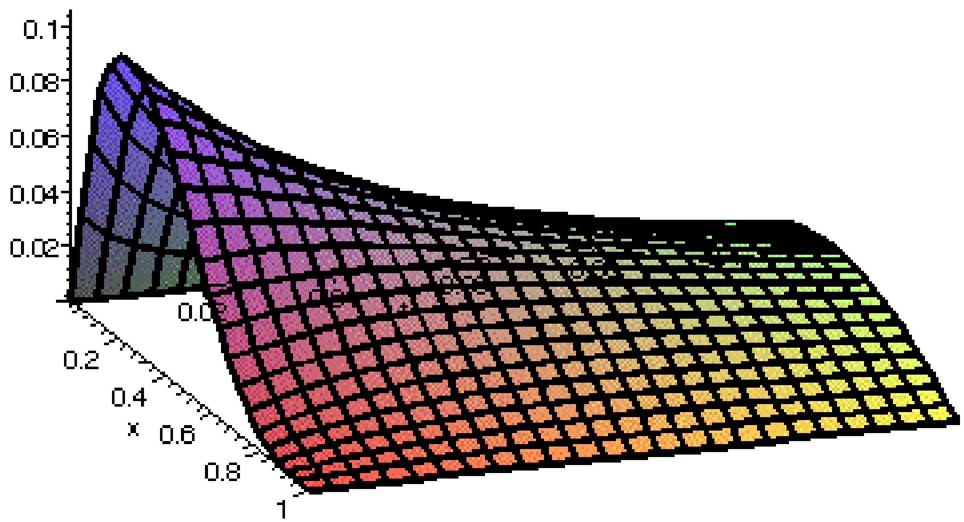
The condition $u(0, x) = f(x)$ determines the c_n's.

$$C_n = 2\int_0^1 f(x)\sin(n \Pi x)\,dx.$$

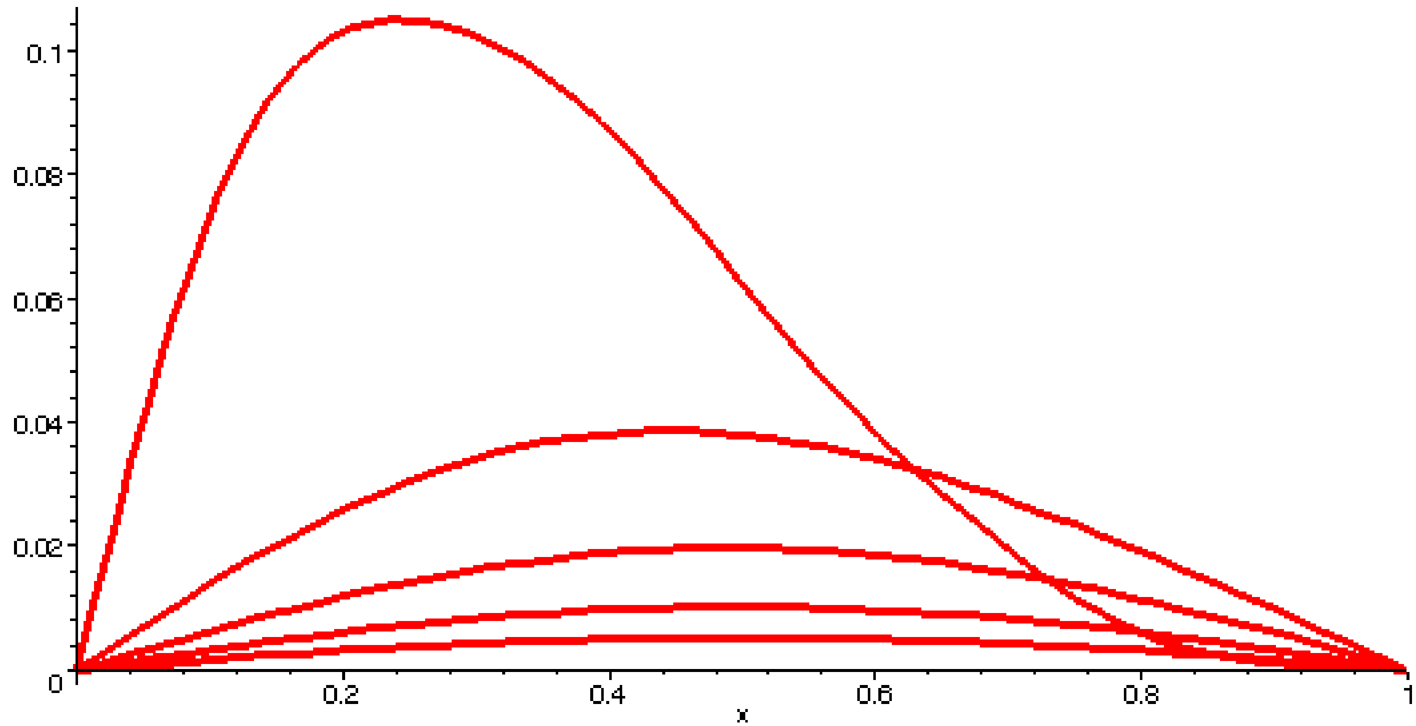
Graph of f(x) = x $(1-x)^3$ and a Fourier approximation. Type convergence?



Graph of u(t, x) with u(0, x) = $x (1-x)^3$



Trace the highpoint:



Assignment: See the Maple worksheet.

In this Module 16, we have solved a simple heat equation with zero boundary conditions and with an initial distribution.