Module 17: A Variation on the Heat Equation

Recall Newton's Law of Cooling:

$$T' = -k\,(T - A), \quad T(0) = \text{specified}$$

has solution

$$T(t) = A + \exp(-k\,t)\,(T(0)-A).$$

A rod is not a point!

Diffusion and radiation cooling:

$$\frac{dZ}{dt} = \frac{d^2 Z}{dx^2} - k(Z - A),\ Z(t,0) = c,\ Z(t,1) = c$$

with $Z(0,x) = f(x)$.

There are two perspectives:

Make the simple heat equation more complicated in several steps.

Make this diffusion, radiation cooling simpler in several steps.

$$(*4)\ \frac{dZ}{dt} = \frac{d^2 Z}{dx^2} - k(Z - A),\ Z(t,0) = c,\ Z(t,1) = c$$

with $Z(0,x) = f(x)$.

$$(*3)\ \frac{dw}{dt} = \frac{d^2 w}{dx^2} - k\,w,\ w(t,0) = c - A,\ w(t,1) = c - A$$

with $w(0,x) = f(x) - A$.

The connection: $w(t, x) = Z(t, x) - A$

or $Z(t, x) = w(t, x) + A$

(*3) $\frac{dw}{dt} = \frac{d^2 w}{dx^2} - k w,\ w(t,0) = c - A,\ w(t,1) = c - A$

with $w(0,x) = f(x) - A.$

(*2) $\frac{dv}{dt} = \frac{d^2 v}{dx^2} - k v,\ v(t,0) = 0,\ v(t,1) = 0$

with $v(0,x) = f(x) - A - s(x).$

The connection:

s solves 0 = s '' - k s, s(0) = c - A = s(1)

w(t, x) = v(t, x) + s(x) or v(t, x) = w(t, x) - s(x)

(*2) $\frac{dv}{dt} = \frac{d^2 v}{dx^2} - k v,\ v(t,0) = 0,\ v(t,1) = 0$

with $v(0,x) = f(x) - A - s(x).$

(*1) $\frac{du}{dt} = \frac{d^2 u}{dx^2},\ u(t,0) = 0,\ u(t,1) = 0$

with $u(0,x) = f(x) - A - s(x).$

The connection: v(t, x) = exp(-k t) u(t, x)

Or u(t, x) = exp(k t) v(t, x)

Solve (*1) for u.

Make up $v(t, x) = \exp(-k\ t)\ u(t, x)$.

Make up s as the solution for

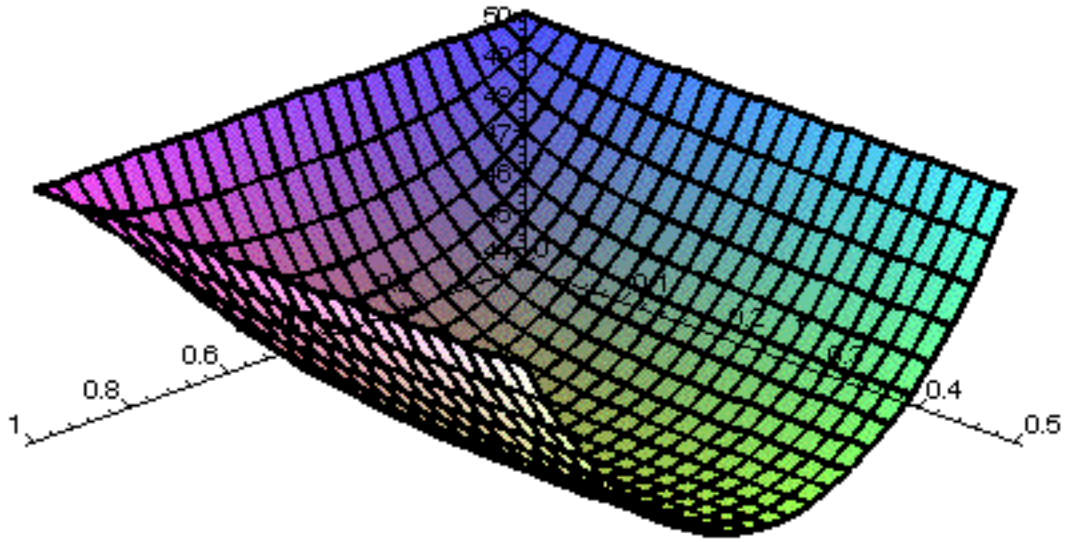
$$0 = s'' - k\ s,\ s(0) = c - A = s(1).$$

Make up $w(t, x) = v(t, x) + s(x)$.

Make up $Z(t, x) = w(t, x) + A$.

$$\frac{dZ}{dt} = \frac{d^2Z}{dx^2} - k(Z-32),\ Z(t,0) = 50,\ Z(t,1) = 50$$

with $Z(0,x) = 50.$



Assignment: See the Maple Worksheet.

In this Module 17, we have suggested that differential equations can often be transformed to simpler ones for which a solution is more easily obtained.