Module 18: Insulated Boundary Conditions

In this module, we begin an examination of the heat equation with different boundary conditions.

PDE: $du/dt = d^2u/dx^2$,

Boundary Conditions:

$du/dx(t, 0) = 0$ and $du/dx(t,1) = 0$.

Initial Condition: $u(0, x) = f(x)$.

Find two ODE's, with boundary conditons,

$$X T' = X'' T,$$

or

$$T' / T = X'' / X.$$

$$X'' = \mu X, \text{ with } X'(0) = 0 \text{ and } X'(1) = 0,$$

and

$$T' = \mu T.$$

$\mu = -n^2\pi^2$ with $X(x) = \cos(n\pi x)$, for $n = 0, 1, \ldots$

Also, $T(t) = \exp(-n^2\pi^2 t)$.

General solution

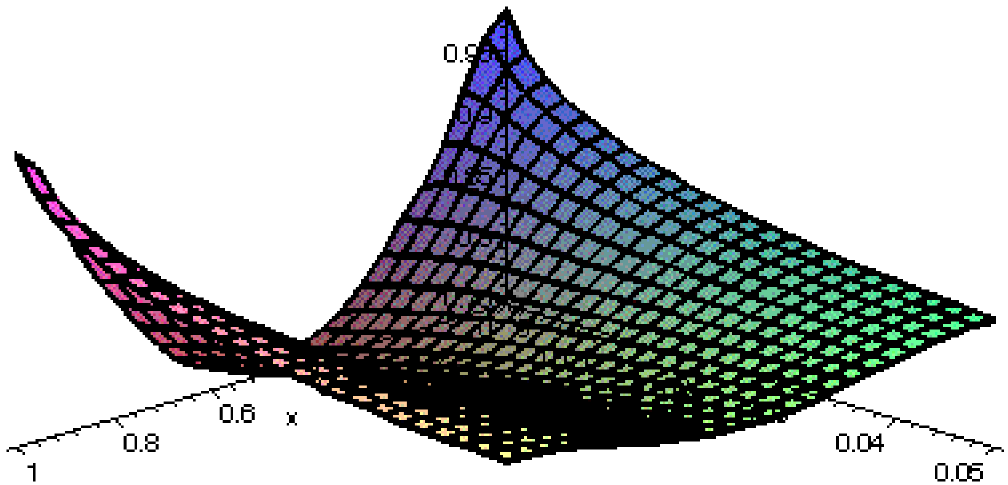
$$u(t,x) = c_0 + \sum c_n \exp(-n^2\Pi^2 t)\cos(n\Pi x)$$

Particular solution, specify f.

$$f(x) = c_0 + \sum c_n \cos(n\Pi x) \ .$$

$$c_n = 2\int_0^1 f(x)\cos(n\Pi x)\, dx$$

Graph when f(x) = 1 + x (x-1).



Second Problem:

PDE: $du/dt = d^2u/dx^2$,

Boundary Conditions:

$u(t, 0) = A$ and $du/dx(t,1) = 0$.

Initial Condition: $u(0, x) = f(x)$.

Physical interpretation?

Take $v(x) = A$ and then

PDE: $du/dt = d^2u/dx^2$,

Boundary Conditions:

$$u(t, 0) = 0 \text{ and } du/dx(t,1) = 0.$$

Deal with the initial conditions later.

The homogeneous problem leads to two ODE 's, one having boundary conditions:

$$X'' = \mu X, \text{ with } X(0) = 0 \text{ and } X'(1) = 0,$$

and

$$T' = \mu T.$$

$\mu = -\lambda^2$, $X = \alpha \cos(\lambda x) + \beta \sin(\beta x)$.

$X(0) = 0$ implies $\alpha = 0$. $X'(1) = 0$ implies

$$\beta \lambda \cos(\lambda) = 0.$$

Thus, $\cos(\lambda) = 0$ so that λ is an odd multiple of $\pi/2$.

That is,

$$\lambda_n = (2n - 1)\,\pi/2\ ,\ n = 1, 2, 3, \ldots\ .$$

These are the eigenvalues corresponding to eigenfunctions $\sin(\lambda_n x)$.

The corresponding solution for the T equation is

$$T(t) = \exp(-\lambda_n^2 t) .$$

The general solution for the original problem:

$u(t, x) + v(x)$

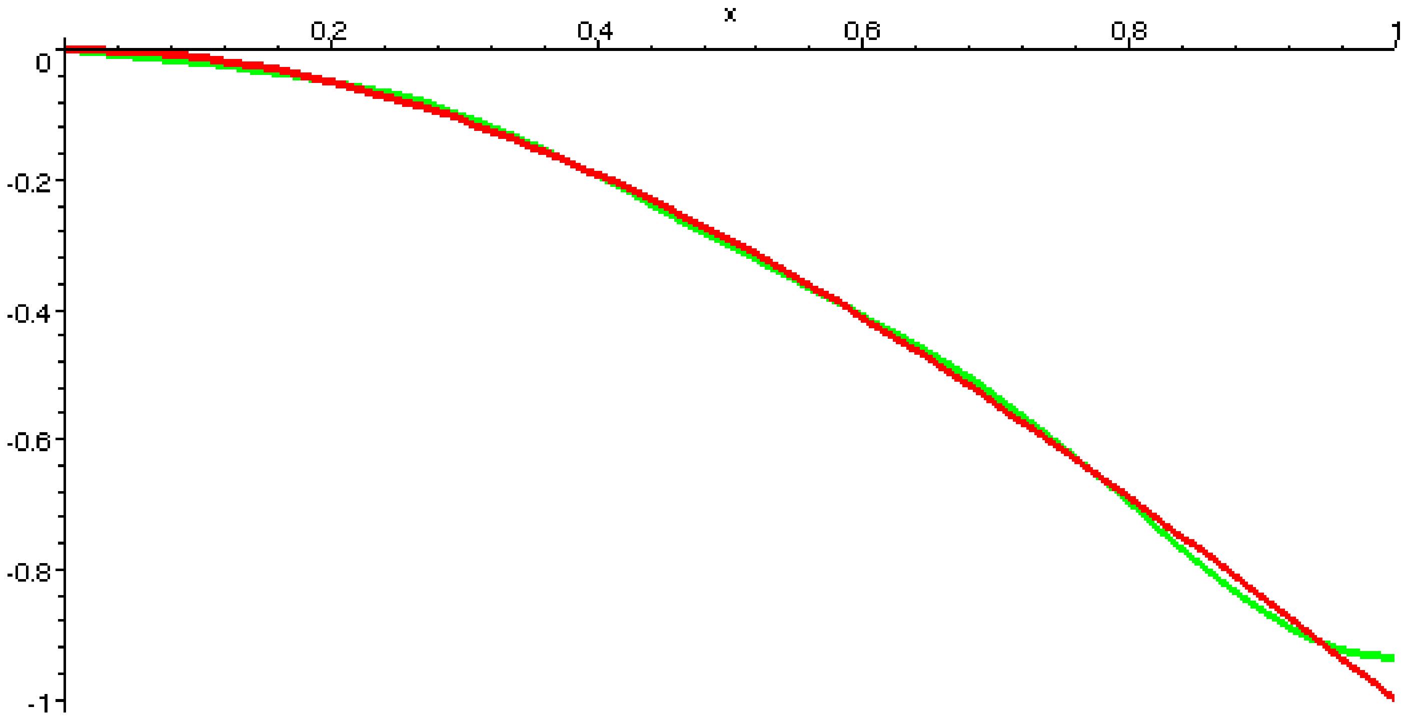
$$= \sum c_n \exp(-(2n-1)^2 \Pi^2 t/4) \sin((2n-1)\Pi/2x) + A$$

To determine the c 's, let t = 0, and use the Fourier idea.

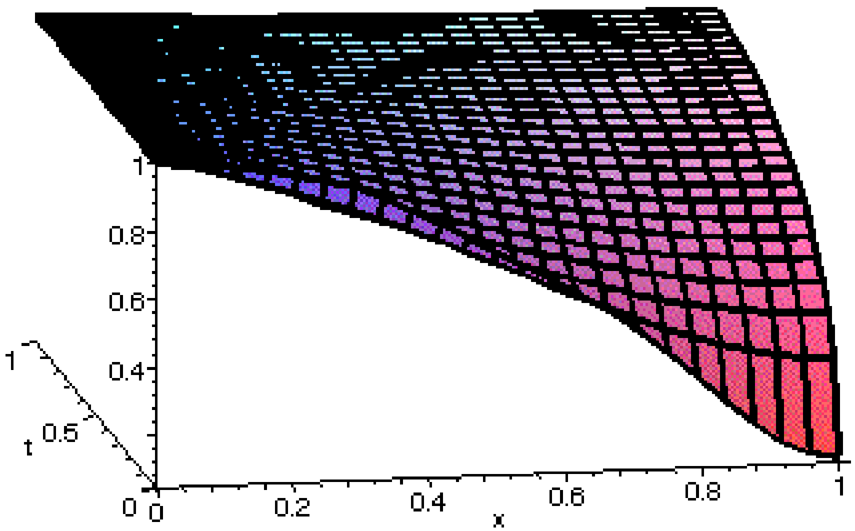
To be specific, we choose a particular f and A.

$f(x) = \cos(\pi/2\ x)$ and $A = 1$.

$$Cn = 2 \int_0^1 (f(x) - A)\sin((2n-1)\Pi/2x)\,dx$$



Graph of A + u(t, x)



Assignment: See Maple Assignment.

In this Module 18, we have solved the heat equation twice:

1. with both ends insulated, and
2. with one end at a constant temperature and the other end insulated.