Module 19: Convection Across Boundaries

PDE: $dw/dt = d^2w/dx^2$,

Boundary Conditions:

$dw/dx(t, 0) = 0$ and $dw/dx(t,1) = -(w(t,1) - 2)$.

The Initial Condition: $w(0, x) = 4\,x\,(1-x) + 2$.

This PDE has a physical interpretation.

The PDE was homogeneous, but the boundary conditions were not.

PDE: $du/dt = d^2u/dx^2$,

Homogeneous Boundary Conditions:

$du/dx(t, 0) = 0$ and $du/dx(t,1) = - u(t,1)$.

Is it clear that the steady state solution is

$v(x) = 2$ and that $u = w - 2$?

Our Job: find the general solution for the PDE with homogeneous boundary conditions.

Separation of variables leads to two ordinary differential equations, one having boundary conditions.

$X'' = -\lambda^2 X$, $X'(0) = 0$, $X'(1) + X(1) = 0$.

$T' = -\lambda^2 T$

$X(x) = A\sin(\lambda x) + B\cos(\lambda x)$.

The first boundary condition implies $A = 0$.

$$X(x) = 0 \sin(\lambda x) + B \cos(\lambda x).$$

$$0 = X'(1) + X(1) = -B \lambda \sin(\lambda) + B \cos(\lambda)$$

or

$$\tan(\lambda) = 1/\lambda .$$

We'll find these λ 's. Then,
Eigenvalues are $\mu = -\lambda^2$ and
eigenfunctions are $\cos(\lambda x)$.

We find a sequence of λ_n 's and make

$$u(t, x) = \sum c_n \exp(-\lambda_n{}^2 t) \cos(\lambda_n x) .$$

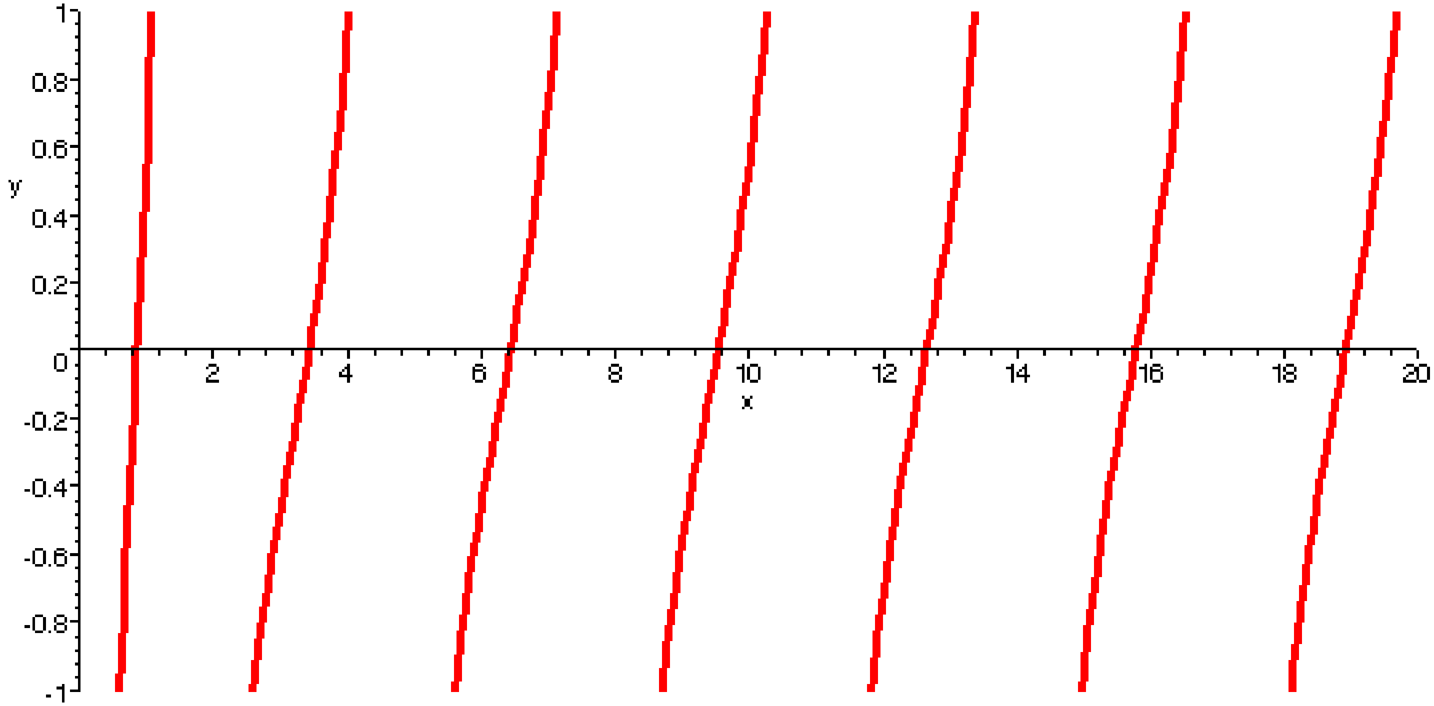
Good luck at trying to get the λ_n 's in closed form! Rather, we need a numerical procedure.

Recall Newton's Method:

To find s so that g(s) = 0, make a first guess and iterate: x_0=first guess,

$$X_{n+1}=x_n - g(x_n)/g'(x_n).$$

Graph of tan(x) - 1/x



$\lambda_1 = 0.860335890$

$\lambda_2 = 3.425618459$

$\lambda_3 = 6.437298179$

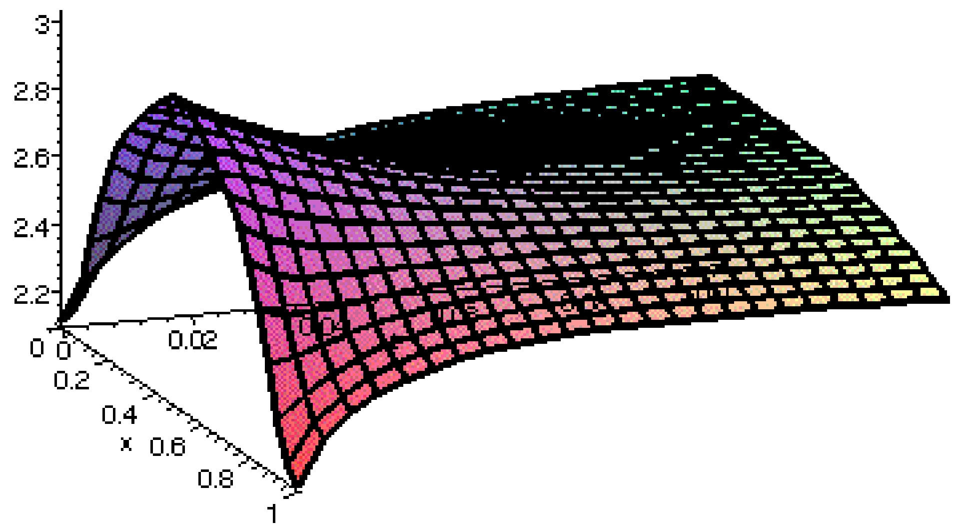
$\lambda_4 = 9.529334405$

...

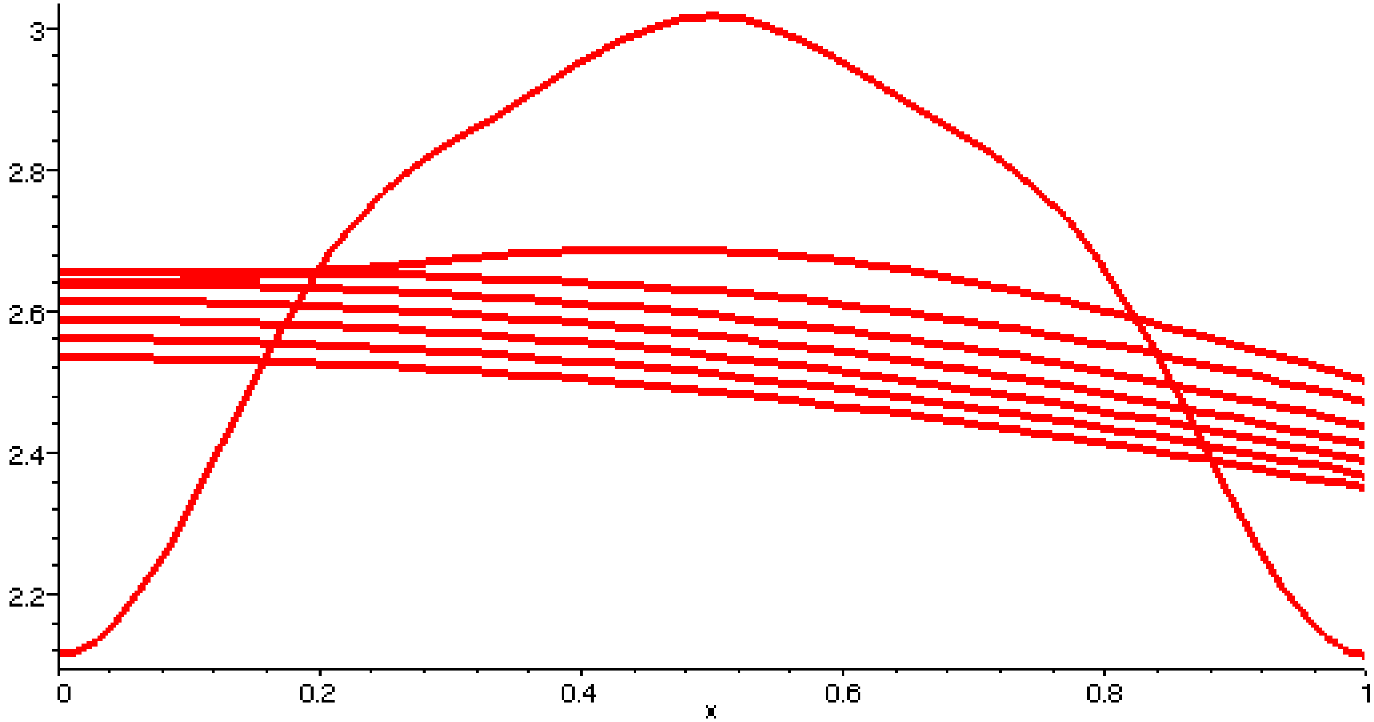
$$u(t, x) = \sum c_n \exp(-\lambda_n^2 t) \cos(\lambda_n x) .$$

$$c_n = \int_0^1 (f(x) - 2) \cos(\lambda_n x)\, dx \Big/ \int_0^1 \cos(\lambda_n x)^2\, dx$$

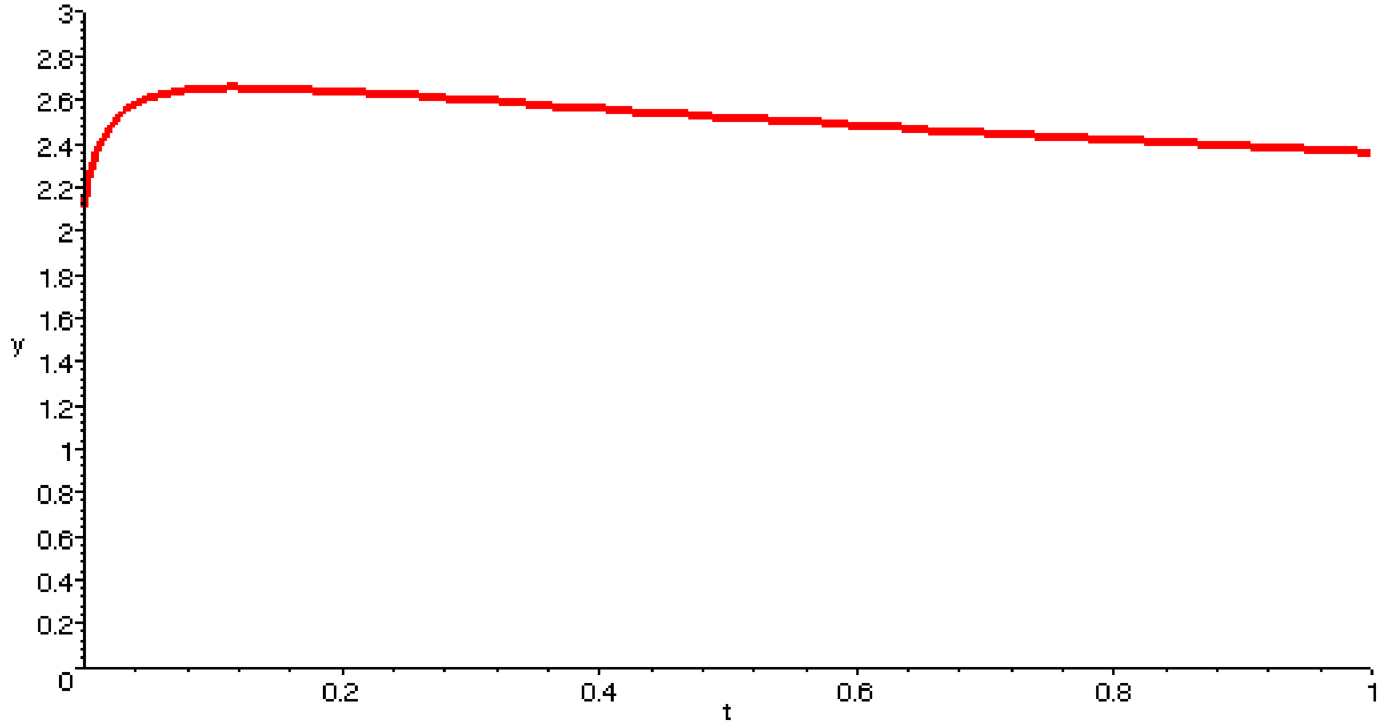
Graph of w(t, x)



Snapshots of w(t, x) as t increases.



Left end temperature as t increases



Assignment: See Maple worksheet

In this Module 19, we have investigated the heat equation with convection boundary conditions.