Module 20: Structure of Solutions

The PDE: $du/dt = d^2u/dx^2$,

The Boundary Conditions: $u(t, 0)=T_0$, $u(t, 1) = T_1$

The Initial Conditions: $u(0, x) = f(x)$,

f is continuous, with $f(0) = T_0$, $f(1) = T_1$

Existence: There is one and only one solution for the problem, and it exists for t in $[0, \infty)$, x in $[0, a]$. Moreover, the solution u is infinitely differentiable for t in $(0, \infty)$, x in $[0, a]$.

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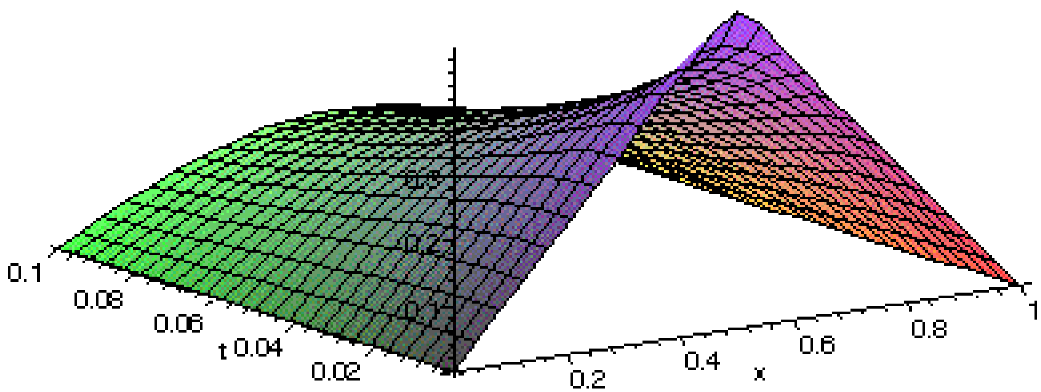
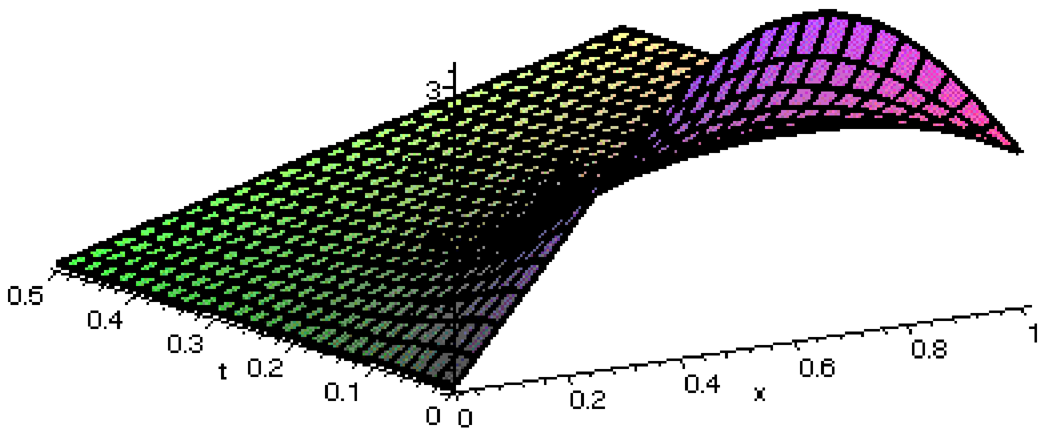
Solutions have an intolerance for sharp corners.



Asymptotic Properties for Solutions: As t increases, u(t, x) approaches a solution which is independent of t. For the above: $(T_1 - T_0)\,x + T_0$.



Take an initial distribution of heat with two or more peaks and see if the heat from one peak might combine with the heat from another peak to produce a temperature greater than that for either of the two initial peaks.

Maximal Principle: The solution for the above heat equation for t in $[0, T]$ and x in $[0, a]$ attains its maximum and minimum on one of three sides of the boundary of the region in the plane.

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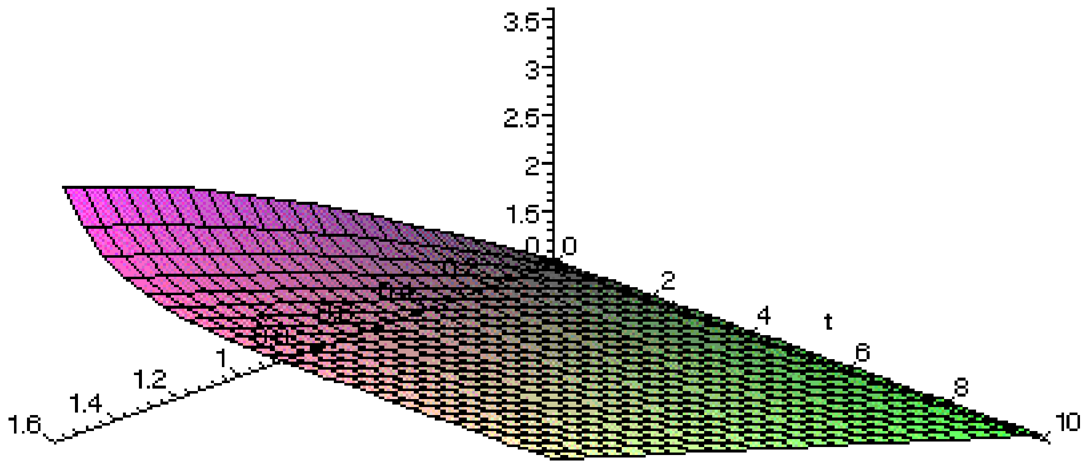
That is, the temperature in the interval [0, a] can never exceed the larger of the maximum of the initial temperature or the maximum temperature at the ends of the interval [0, a] as time increases.

What is the maximum value of $u(t, x)$ and where does it occur for $\exp(-t)\sin(x) + x + 1$ on $[0, \pi/2]$?



Suppose that u satisfies the differential equation above and that v satisfies the equation

The PDE: $dv/dt = d^2v/dx^2$,
The Boundary Conditions: $v(t, 0)=T_0$, $v(t, 1) = T_1$

The Initial Conditions: $v(0, x) = g(x)$

Suppose also that $f \leq g$. We argue that for all t and x in the appropriate intervals, $u(t, x) \leq v(t, x)$.

Let $w(t, x) = u(t, x) - v(t, x)$ and ask what equation w satisfies.

The PDE: $dw/dt = d^2w/dx^2$,

The Boundary Conditions: $w(t, 0)=0$, $w(t, 1) = 0$,

The Initial Conditions: $w(0, x) = f(x) - g(x)$.

Conclude that $w(t, x) \leq 0$.

Assignment: See Maple Worksheet.

In this Module 20, we have discussed

1. Existence and Uniqueness of Solutions,
2. Smoothness of Solutions,
3. Maximum Principle for the Heat Equation,
4. Order properties for Pairs of Solutions.