Module 21: The Wave Equation in One Dimension

The classical, linearized wave equation is

$$d^2u/dt^2 = c^2\ d^2u/dx^2.$$

The more general constant coefficient, second order equation:

$$a\ d^2u/dt^2 + b\ d^2u/dtdx + c\ d^2u/dx^2 + d\ du/dt + e\ du/dx + f = 0.$$

Boundary Conditions: $u(t, 0) = 0$, and $u(t, L) = 0$.

Because the equation is second order in t, we expect two initial conditions.

Initial Conditions: $u(0, x) = f(x)$, $du/dt(0, x) = g(x)$ for x in $[0, L]$.

Physical realization for this model is the vibrations of a taut string.

Separation of variables leads to

$$T''/T = X''/X.$$

Boundary conditons lead to

$$X'' = -\lambda^2 X, \quad X(0) = 0 = X(L) \text{ and}$$
$$T'' = -\lambda^2 T.$$

$\lambda = n^2 \pi^2 /L^2$, $X(x) = \sin(n \pi x/L)$, and
$T(t) = \sin(n \pi x/L)$ or $\cos(n \pi x/L)$.

General Solution:

$$U(t, x) = \sum (A_n \cos(n\Pi t/L) + B_n \sin(n\Pi t/L))\sin(n\Pi x/L)$$

$$f(x) = u(0, x) = \sum A_n \sin(n\Pi x/L)$$

$$g(x) = u_t(0, x) = \sum B_n \frac{n\Pi}{L} \sin(n\Pi x/L)$$

$$\int_0^L \sin(n\Pi x/L)^2\,dx = L/2\ ,$$

$$f(x) = \sum A_n \sin(n\Pi x/L)$$

$$A_n = 2/L \int_0^L f(x)\sin(n\Pi x/L)dx\ ,$$

$$g(x) = \sum B_n \frac{n\Pi}{L} \sin(n\Pi x/L)$$

$$B_n = 2/n\Pi \int_0^L g(x)\sin(n\Pi x/L)dx\,.$$

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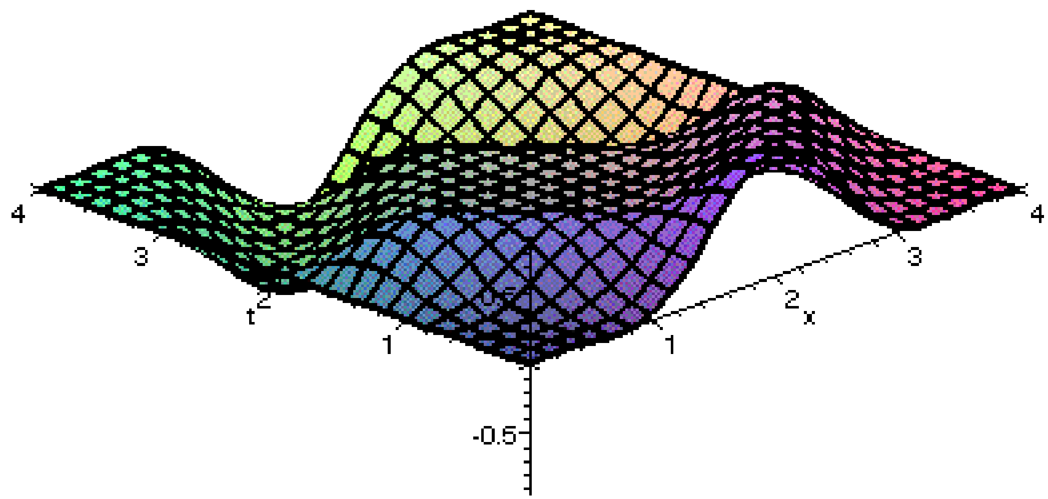
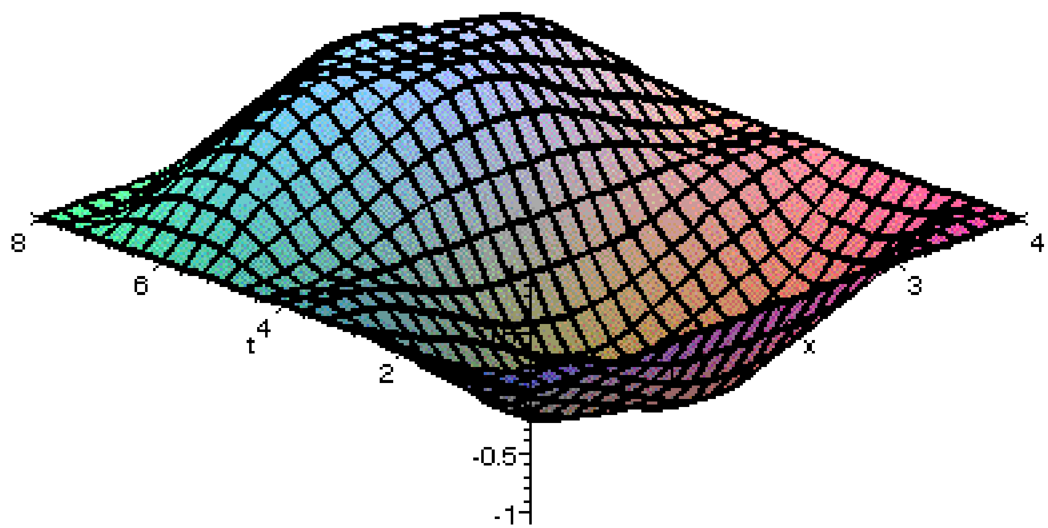


f(x) = 0, g(x) = a whack



Assignment: See Maple worksheet

In this Module 21, we have

1. Introduced the simple, one dimensional wave equation,
2. Solved this equation by separation of variables, and
3. Graphed two specific solutions.