Module 22: d'Alembert's Solution

$$d^2u/dt^2 = c^2\ d^2u/dx^2$$

Series Solutions or d'Alembert's Solution?

A little history on the matter.

The conceptual difficulty in

(1) having the idea of an infinite sum of functions,

(2) having a series of cosine functions converge to the sine function, and

(3) having the series converge to an analytic function on a finite interval, but not to the function off that interval.

Transforming the Equations.

$$d^2u/dt^2 = c^2\, d^2u/dx^2$$

$u(t, x)$

$t = (w - z)/2$ and $x = (w + z)/2$

or $w = t + x$ and $z = x - t$

$v(w, z)$

$$d^2v/dwdz = 0$$

A solution for the wave equation independent of the Fourier Idea.

$$d^2v/dwdz = 0$$

$$dv/dw = C(w)$$

$$v(w, z) = \Psi(w) + \Phi(z)$$

$$u(t, x) = \Psi(x + t) + \Phi(x - t).$$

Initial conditions for the wave equation:

$$u(0,x) = f(x), \quad u_t(0,x)=g(x)$$

$$u(t, x) = \Psi(x + c\ t) + \Phi(x - c\ t).$$

The initial conditions are that

$$f(x) = u(0, x) = \Psi(x) + \Phi(x)$$

and

$$g(x) = u_t(0, x) = c\ \Psi'(x) - c\ \Phi'(x).$$

$g(x) = u_t(0, x) = c\,\Psi'(x) - c\,\Phi'(x).$

Get an antiderivative:

$G(x) + C = \Psi(x) - \Phi(x).$

$f(x) = \Psi(x) + \Phi(x)$

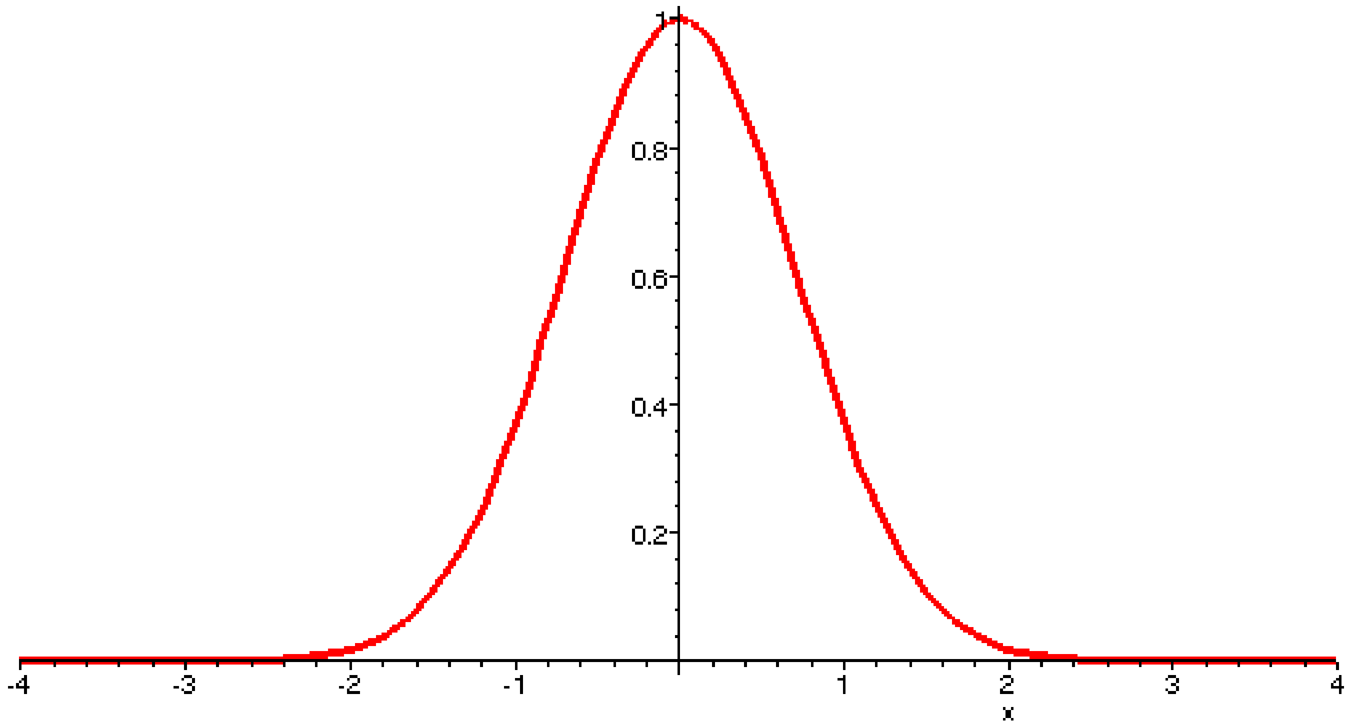
$$G(x) = \frac{\int_0^x g(y)\,dy}{c}$$

$\Psi(x) = (f(x)+G(x)+C)/2$ and $\Phi(x) = (f(x)-G(x)-C)/2$

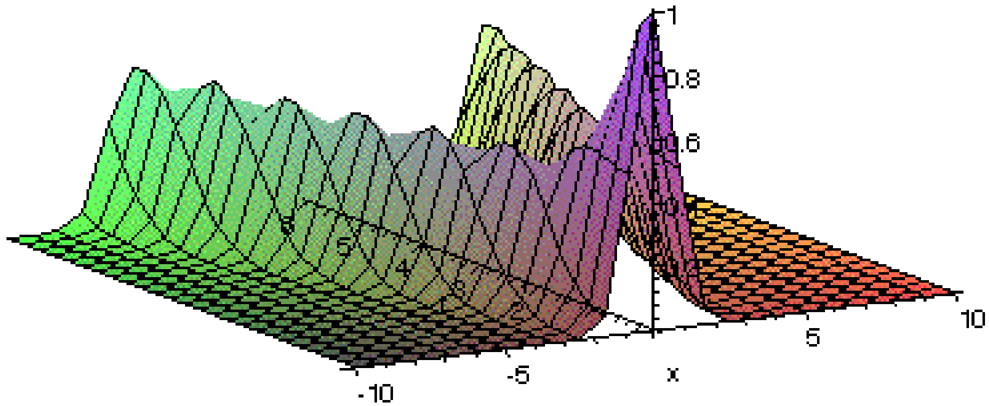
$u(t, x) = (f(x+ct)+f(x-ct))/2 + (G(x+ct)-G(x-ct))/2$

f is a function with one bump near $x = 0$, no initial velocity.

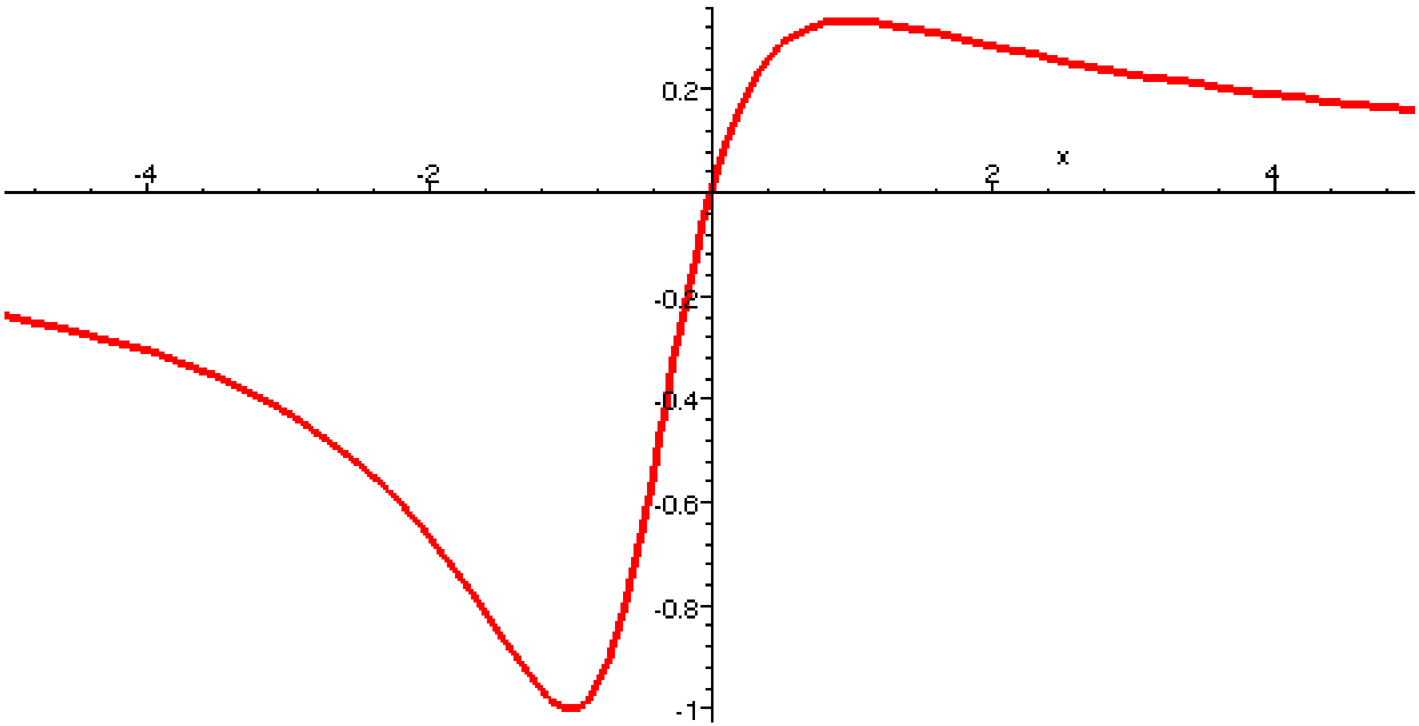
Take, for example, $f(x)=\exp(-x^2)$.



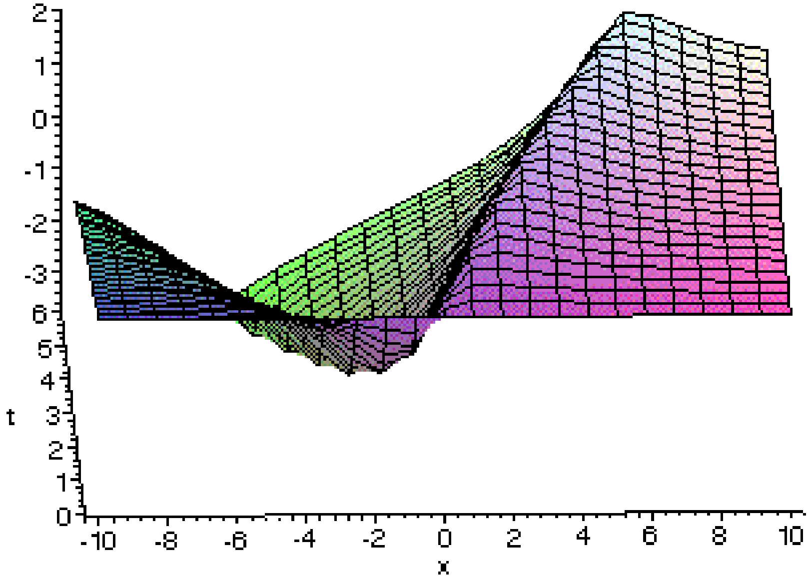
U(t, x), where f(x)=exp(- x^2)



Take f(x) = 0 and g(x) = x/(1+x+x^2).



Graph of u(t, x).



Assignment: See the Maple worksheet.

In this Module 22, we have derived the d'Alembert solution by performing a change of variables on the wave equation. Using this solution, we worked two examples.