Module 23: d'Alembert's Solution for the Wave Equation

d'Alembert's solution has the form

$$u(t, x) = \Psi(x + c\ t) + \Phi(x - c\ t).$$

Solution for any real x and all $t > 0$, with initial conditions

$$u(t, x) = [\ f(x + c\ t) + f(x - c\ t)]/2 + \int_{x-ct}^{x+ct} g(s)\,ds/2$$

Half Infinite Strings

Suppose we solve the equation for $x > 0$. Impose one boundary condition.

Boundary Condition: $u(t, 0) = 0$.

Initial Conditions: $u(0, x) = f(x)$ and $u_t(0, x) = g(x)$ for $x > 0$.

$$u(t, x) = [\, f(x + c\, t) + f(x - c\, t)]/2 + [\, G(x + c\, t) - G(x - c\, t)]/2$$

$$u(t, x) = [\, f(x + c\,t) + f(x - c\,t)]/2 + [\, G(x + c\,t) - G(x - c\,t)]/2$$

where G is an antiderivative of g.

Boundary condition:

$$0 = u(t, 0) = (f(c\,t)+G(c\,t))/2 \;+\; (f(-c\,t)-G(-c\,t))/2$$

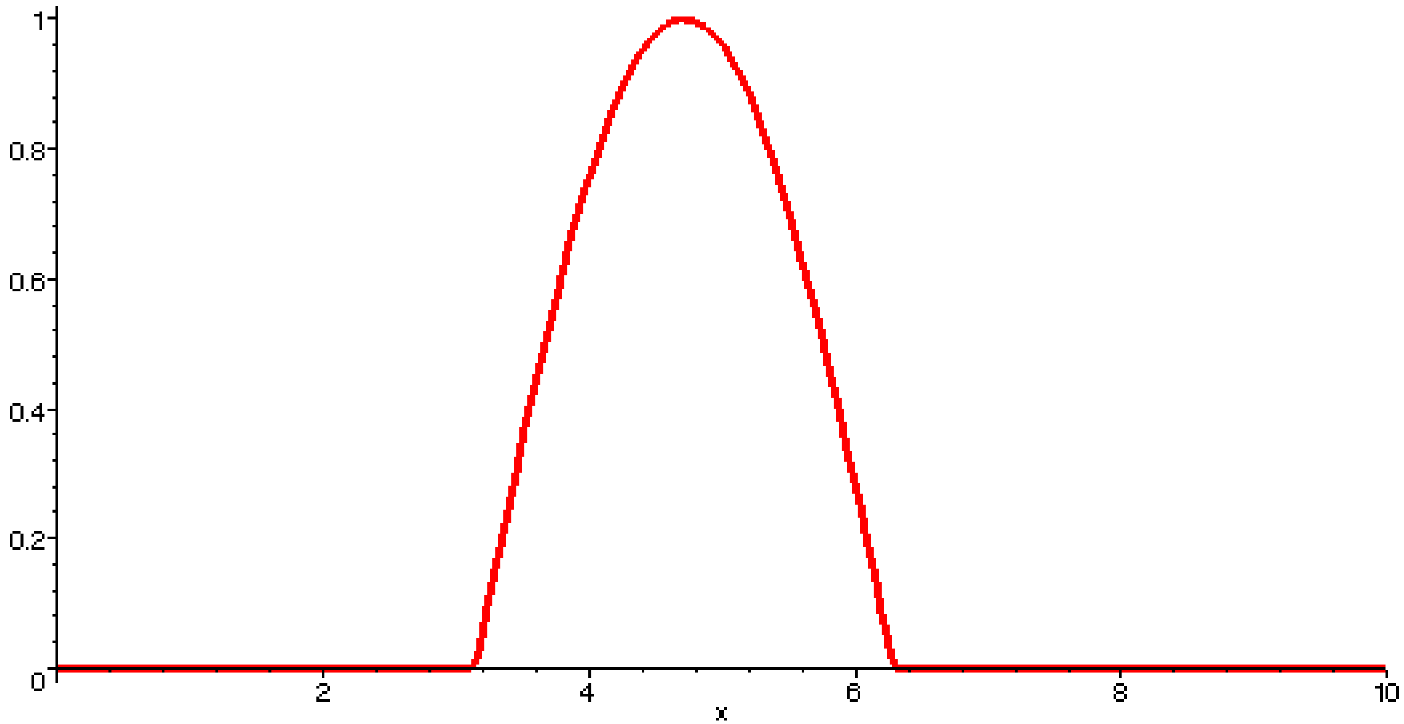
$$0 = f(\,c\,t\,) + f(\,-\,c\,t\,) + G(\,c\,t\,) - G(\,-\,c\,t\,).$$

$f(\,c\,t\,) = -\,f(\,-\,c\,t\,)$ and $G(\,c\,t\,) = G(\,-\,c\,t\,)$

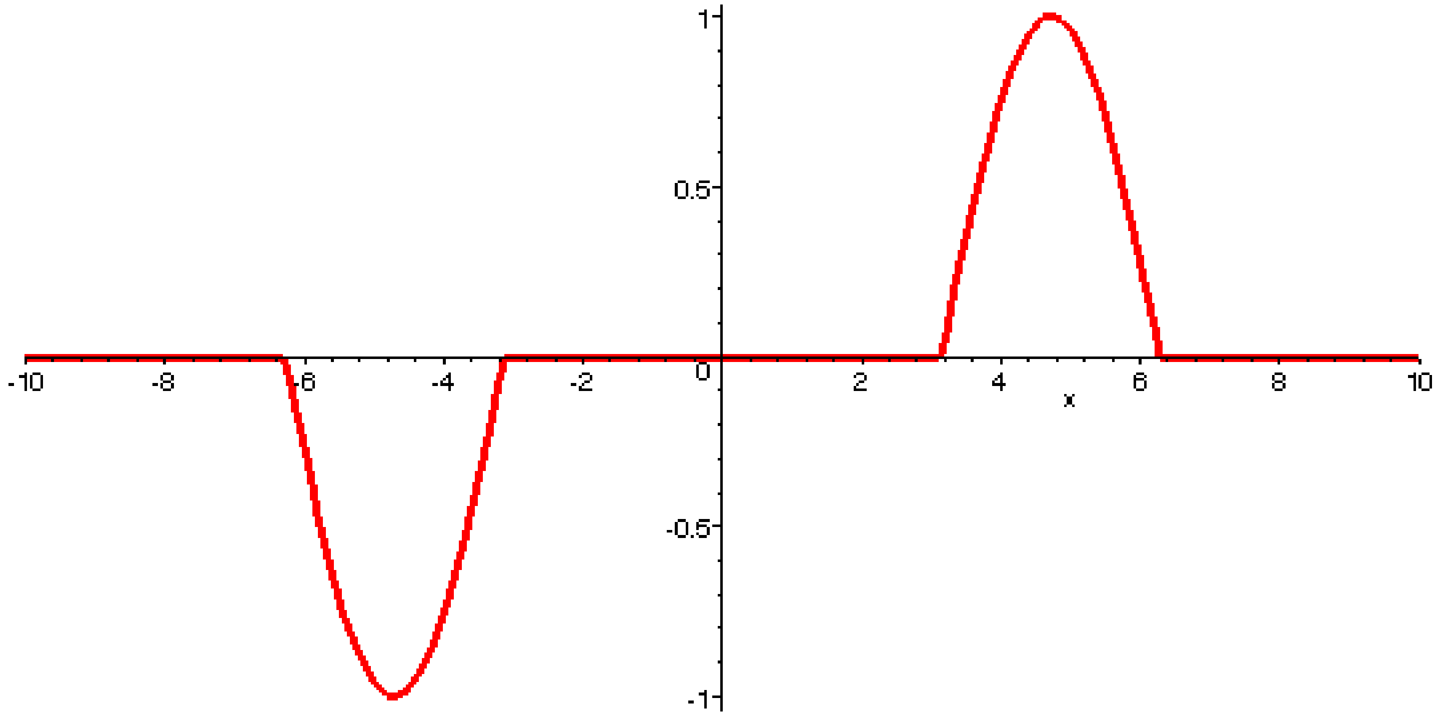
$f(ct) = -f(-ct)$ and $G(ct) = G(-ct)$

f has an odd extension, G even

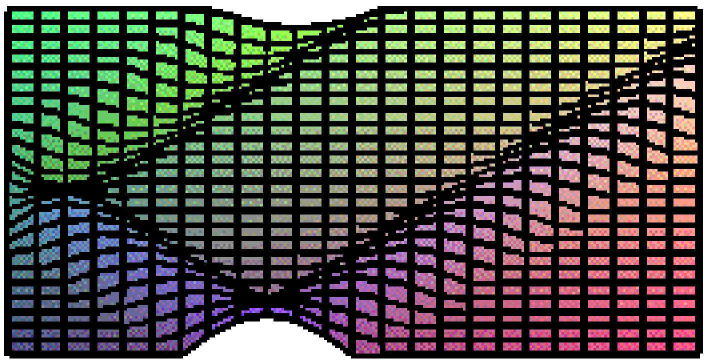
A Problem: Making even and odd extensions?



A Problem: Making even and odd extensions?



Consider contrasts.



Finite Strings

End point at x = 0 and at x = L.

$$0 = u(t, L) = f(L + ct) + f(L - ct) + G(L + ct) - G(L - ct),$$

$$f(L + ct) = -f(L - ct) \quad \text{and}$$
$$G(L + ct) = G(L - ct).$$

f odd, G even

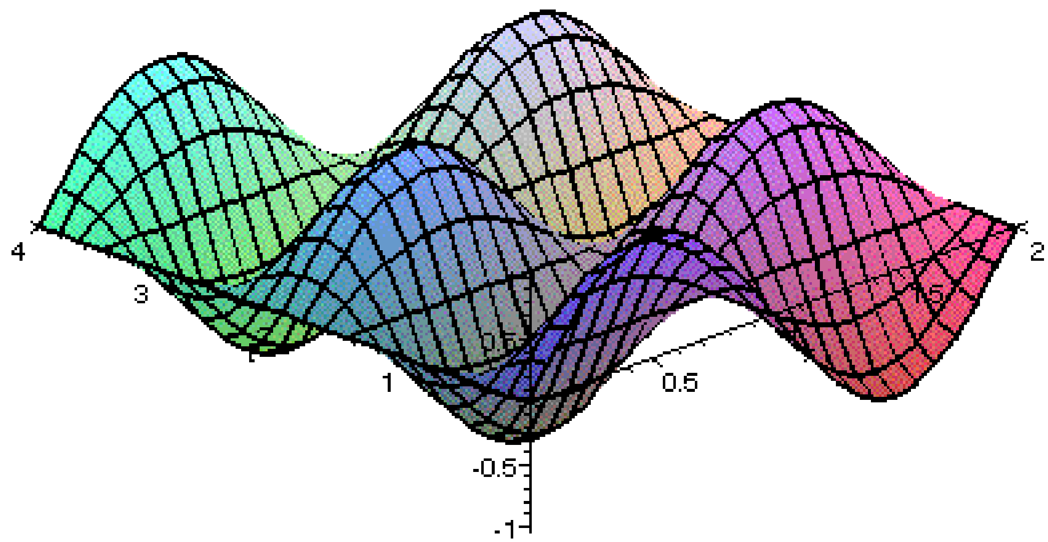
$$f(L + ct) = -f(L - ct) \quad \text{and}$$
$$G(L + ct) = G(L - ct).$$

f odd, G even

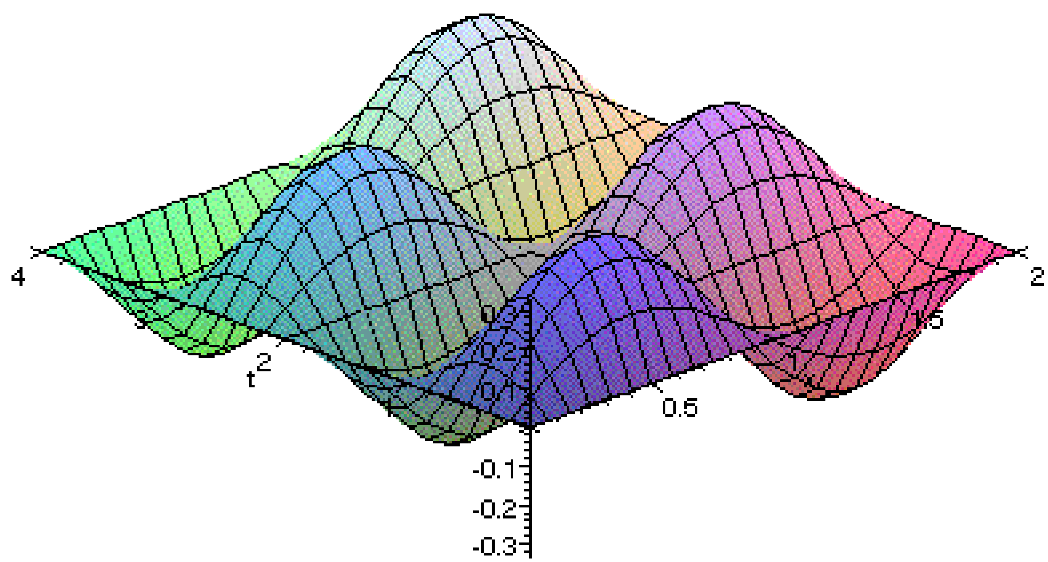
$$f(L + ct) = f(-L + ct) \quad \text{and}$$
$$G(L + ct) = G(-L + ct) \text{ for all } t > 0.$$

That is, f and G have period 2 L. The job: Make odd, 2 L periodic extensions of f and even, 2 L periodic extensions of g.

Example: $c = 1$, $L = 2$, $f(x) = \sin(\pi x)$, $g(x) = 0$.



Example: c=1, L = 2, f(x) = 0, g(x) = sin(π x).



Assignment: See Maple Worksheet.

In this Module 23, we have solved

1. the wave equation for all x,
2. the wave equation for $0 < x$, and
3. the wave equation for $0 < x < L$.