Module 25: A string in a viscous medium

In this worksheet, we suppose that a string is subject to gravity and embedded in a viscous medium.

The problem: find w to satisfy

$$w_{xx} - k\, w_t - g = w_{tt}$$

$$w(t, 0) = 0 = w(t, L)$$

$$w(0, x) = F(x) \quad \text{and} \quad w_t(0, x) = G(x).$$

Simplification: v and w related by

$$v(t, x) = w(t, x) + g\, x\, (L-x)/2$$

or

$$w(t, x) = v(t, x) - g\, x\, (L-x)/2$$

$$v_{xx} - k\, v_t = v_{tt}$$

$$v(t, 0) = 0 = v(t, L)$$

$$v(0, x) = F(x) + g\, x\, (L - x)/2$$

$$v_t(0, x) = G(x).$$

Simplification: u and v related by

$$u(t, x) = \exp(t\,k/2)\, v(t, x)$$

or

$$v(t, x) = \exp(-t\,k/2)\, u(t, x)$$

$$u_{xx} + k^2 u/4 = u_{tt}$$

$$u(t, 0) = 0 = u(t, L)$$

$$u(0, x) = v(0, x)$$

$$u_t(0, x) = k/2\, v(0,k) + v_t(0,x)$$

Solve for u, create v, create w

$$w(t, x) = \exp(-k/2\ t)\, u(t, x) - g\, x\, (L-x)/2$$

$$u_{xx} + k^2 u/4 = u_{tt}$$

$$u(t, 0) = 0 = u(t, L)$$

With k small, this leads to

$$X'' = -\alpha^2 X,\ X(0) = 0 = X(L)$$

$$-\alpha^2 = -(\mu^2 + k^2/4) = -n^2 \pi^2/L$$

$$T'' = -\mu^2\ T$$

$\mu^2 = n^2\pi^2/L - k^2/4$

$T(t)$ is $\sin(\mu t)$ or $\cos(\mu t)$

$$U(t, x) = \sum [A_n \sin(\mu t) + B_n \cos(\mu t)] \sin(n\pi x/L)$$

Determine coefficients.

$w(t, x) = \exp(-k/2\ t)\ u(t, x) - g\ x\ (L-x)/2$

Example:

$$w_{xx} - 1/4 \; w_t = w_{tt}$$

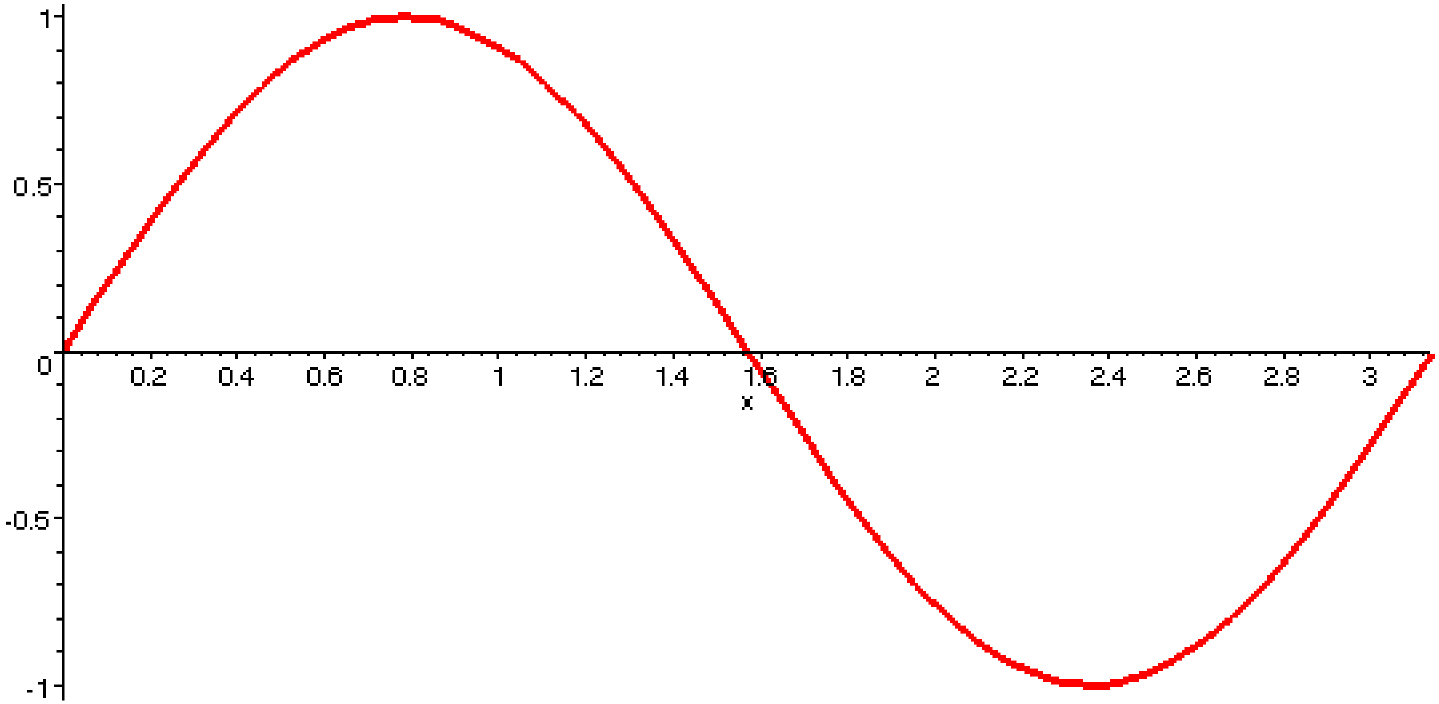
$$w(t, 0) = 0 = w(t, \pi)$$

$$w(0, x) = \sin(2x) \quad \text{and} \quad w_t(0, x) = 0.$$

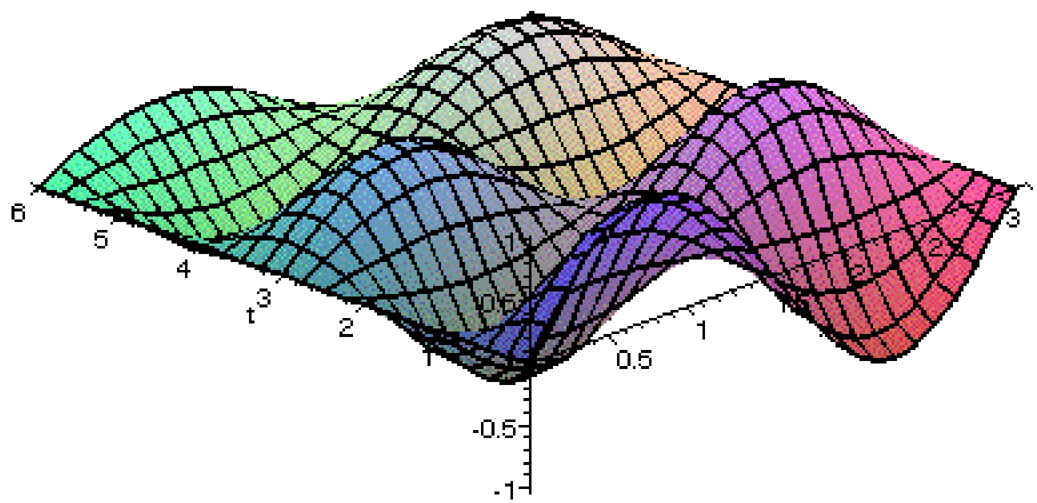
$$w(t, x) = \exp(-k/2\; t)\, u(t, x)$$

$$u(t, x) = \sin(2x)\cos(\sqrt{4-k^2/4}\ t) + k/(2\sqrt{4-k^2/4})\sin(2x)\sin(\sqrt{4-k^2/4}\ t)$$

$w(t, x) = \exp(-k/2\ t)\ u(t,x)$. Initial graph:



Graph of u(t, x)



Assignment: See Maple worksheet.

In this Module 25, we have modeled a string in a viscous medium, subject to gravity.