Module 26: Different Boundary Conditions

A variety of boundary conditions.

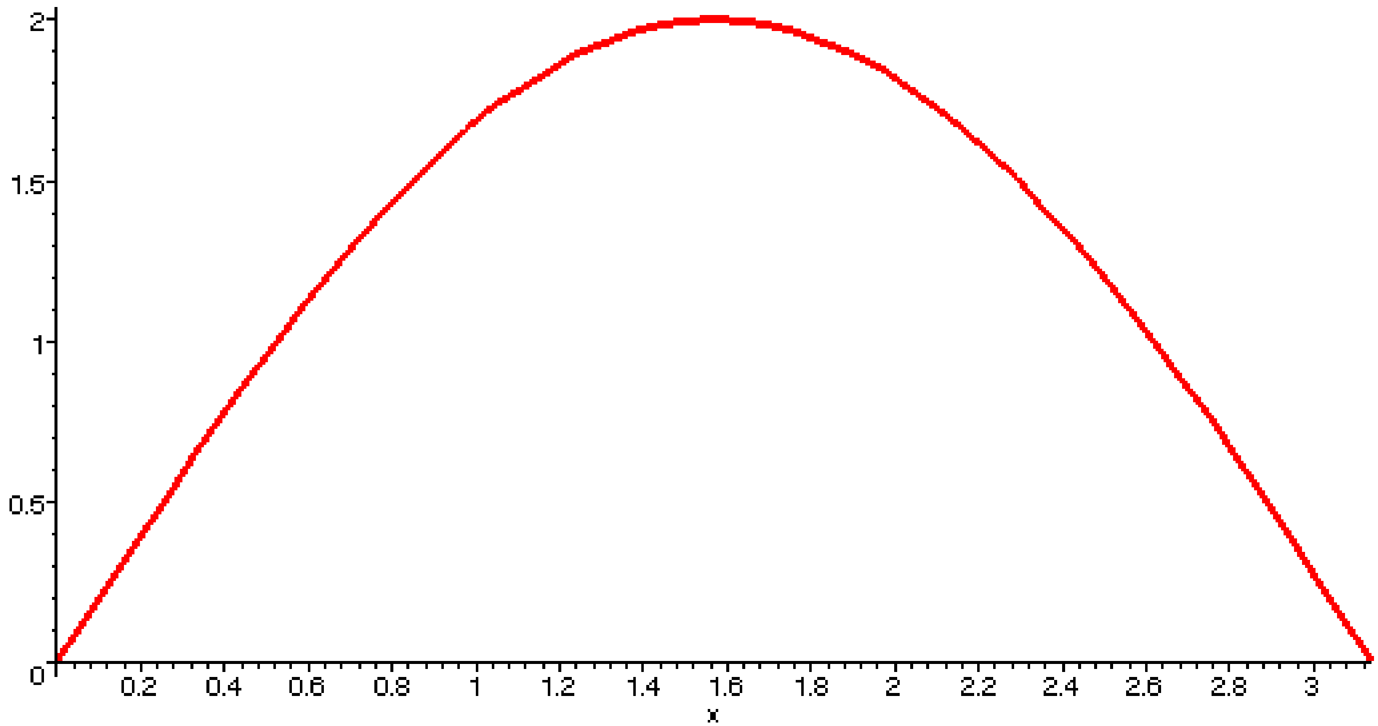
With the heat equation, we had

fixed temperature at the ends,
insulated at the ends, and
Newton's Law of cooling at the ends.

There are similar ideas here.

Fixed endpoints.

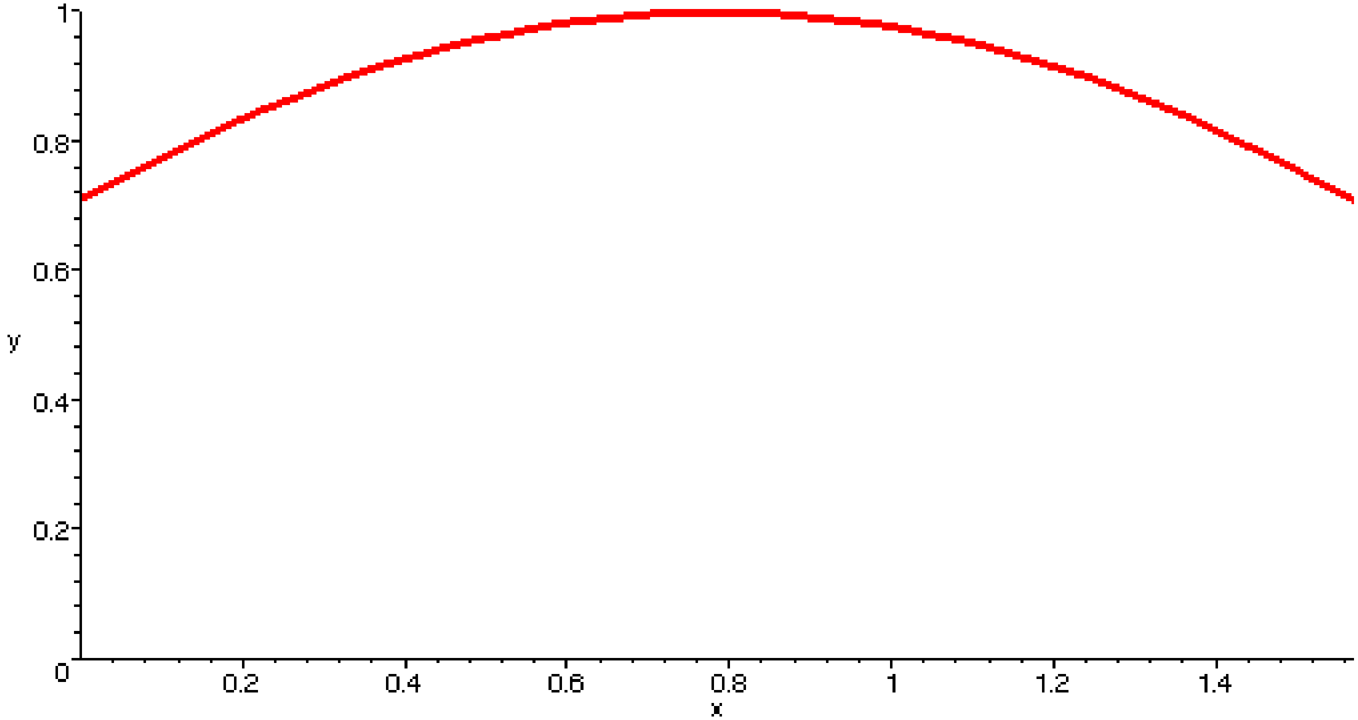
$u(t, 0) = 0$ and $u(t, L) = 0$.



Elastic attachment.

A spring, or other elastic device attached.

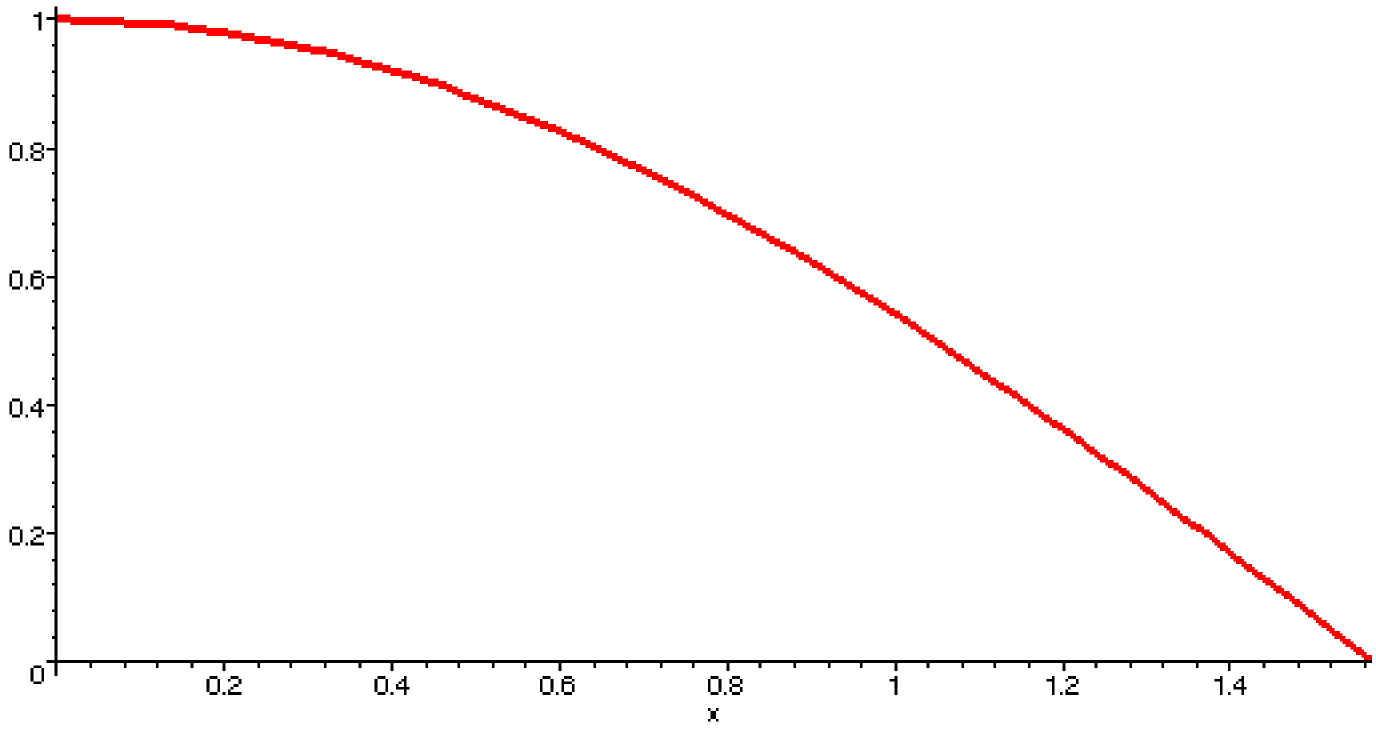
$$u_x(t, 0) = k\, u(t, 0) \quad \text{and} \quad u_x(t, L) = -k\, u(t, L).$$



Frictionless sleeve.

Think carts: $u_x(t, 0) = 0$ and $u_x(t, L) = 0$.

In this illustration, one end is fixed.



Changing Boundary Conditions.
We suppose we have a half infinite string in this model. Change the boundary condition with time.

Initial conditions are zero: $u(0,x) = 0 = u_t(0, x)$.
Boundary condition: $U(t, 0) = b(t)$.
Expect to see a signal move down the string.
Analysis of the problem:

$$U(t, x) = \Psi(x + c\ t) + \Phi(x - c\ t).$$

Using that the initial conditions are zero for $x > 0$, recall that we can conclude that $\Psi(x) = 0$ and $\Phi(x) = 0$ for $x > 0$. The term $\Psi(x + c\, t)$ will be zero since $c > 0$ and $t > 0$. Thus, what ever happens,

$$u(t, x) = \Phi(x - c\, t).$$

and this is zero as long as $x > c\, t$. This result reminds us of the speed of transmission. What happens after $x < c\, t$?

We ask how to extend Φ to the negative numbers.

The answer must lie in the boundary condition. We know that $u(t, 0) = b(t)$, so

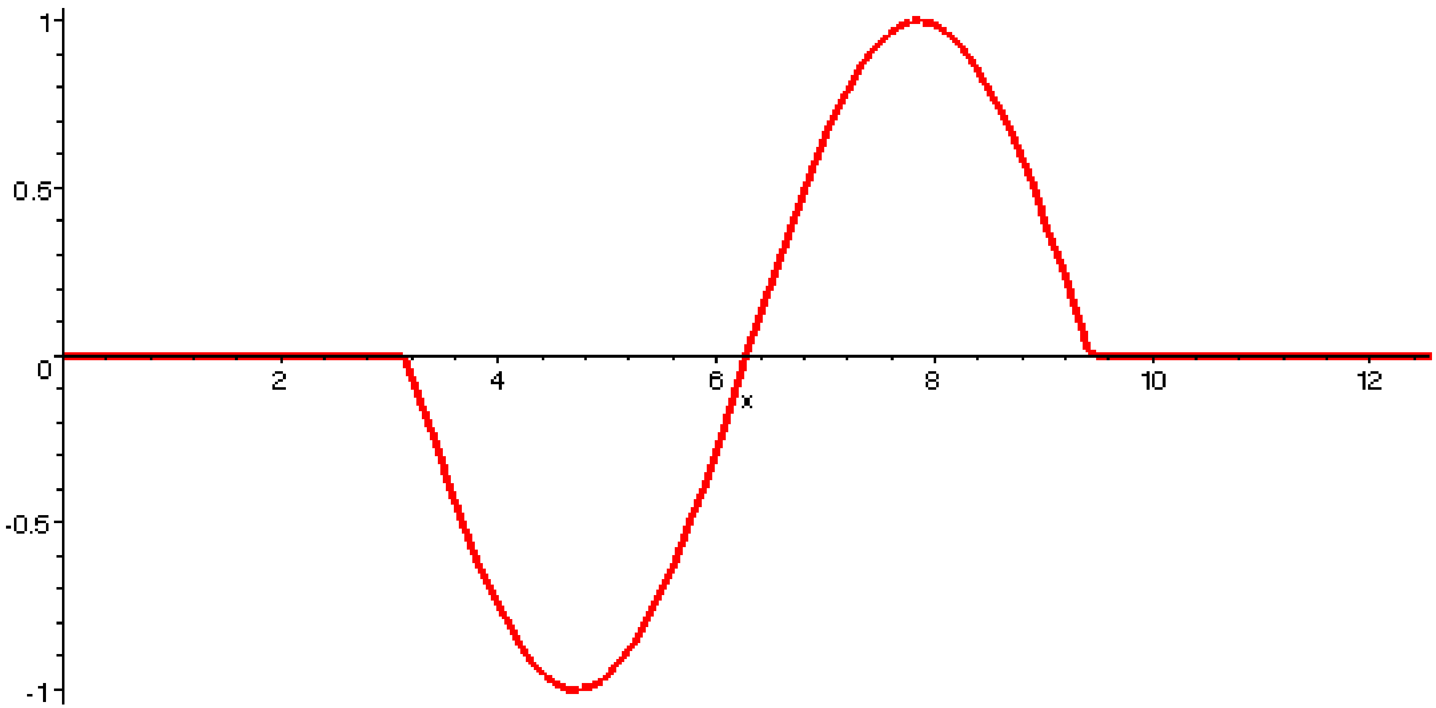
$$\Phi(0 - c\ t) = u(t, 0) = b(t).$$

So, for negative numbers n, we have

$$\Phi(n) = b(-n/c).$$

$U(t, x) = 0$ if $x > c\ t$ and $U(t, x) = b(t - x/c)$ if $x < c\ t$.

Sending a signal:

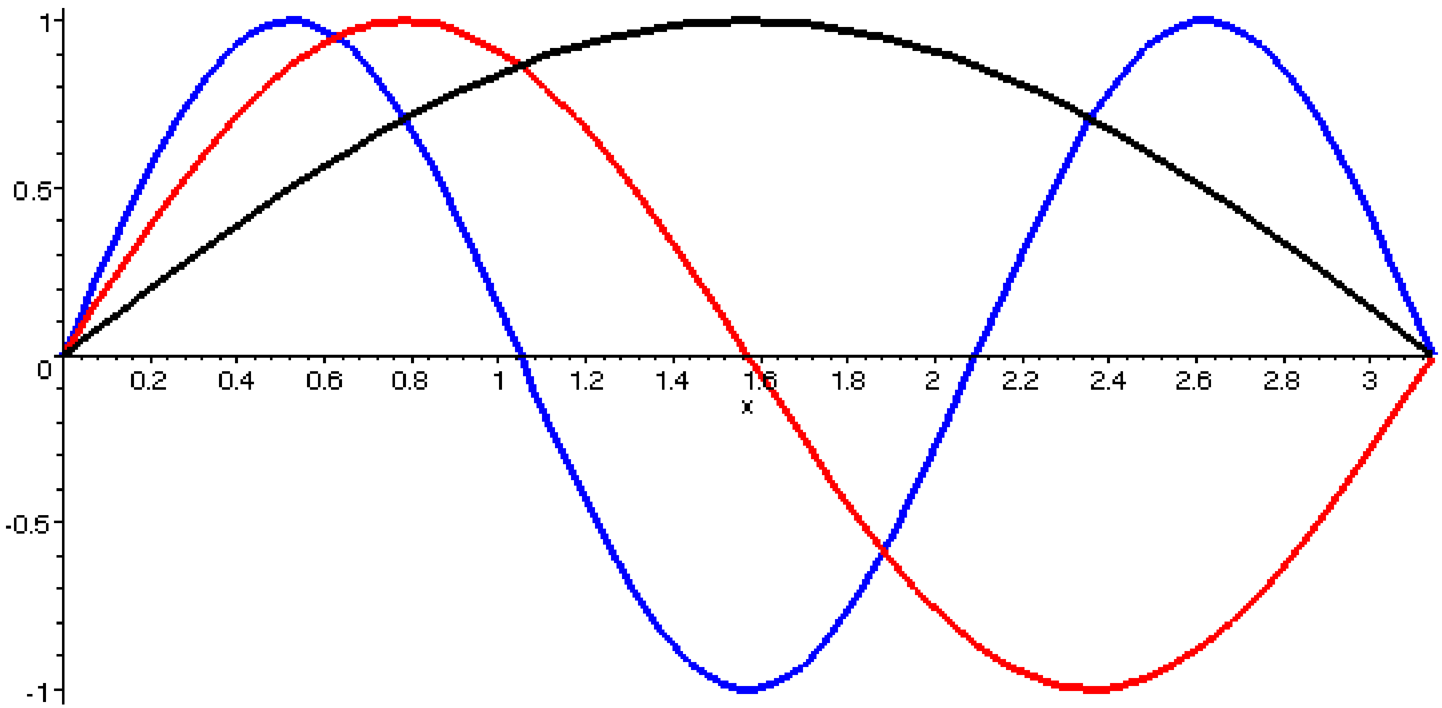


Assignment: See Maple worksheet.

In this Module 26, we have considered four type boundary conditions we might have for a string:
(1) Fixed endpoints,
(2) Elastic attachments,
(3) Frictionless sleeve,
(4) Changing Boundary Condition.

We say a word about Standing Waves and Music.

Standing waves



Solutions for the wave equation.

sin(x) cos(t), or sin(2 x) cos(2 t),

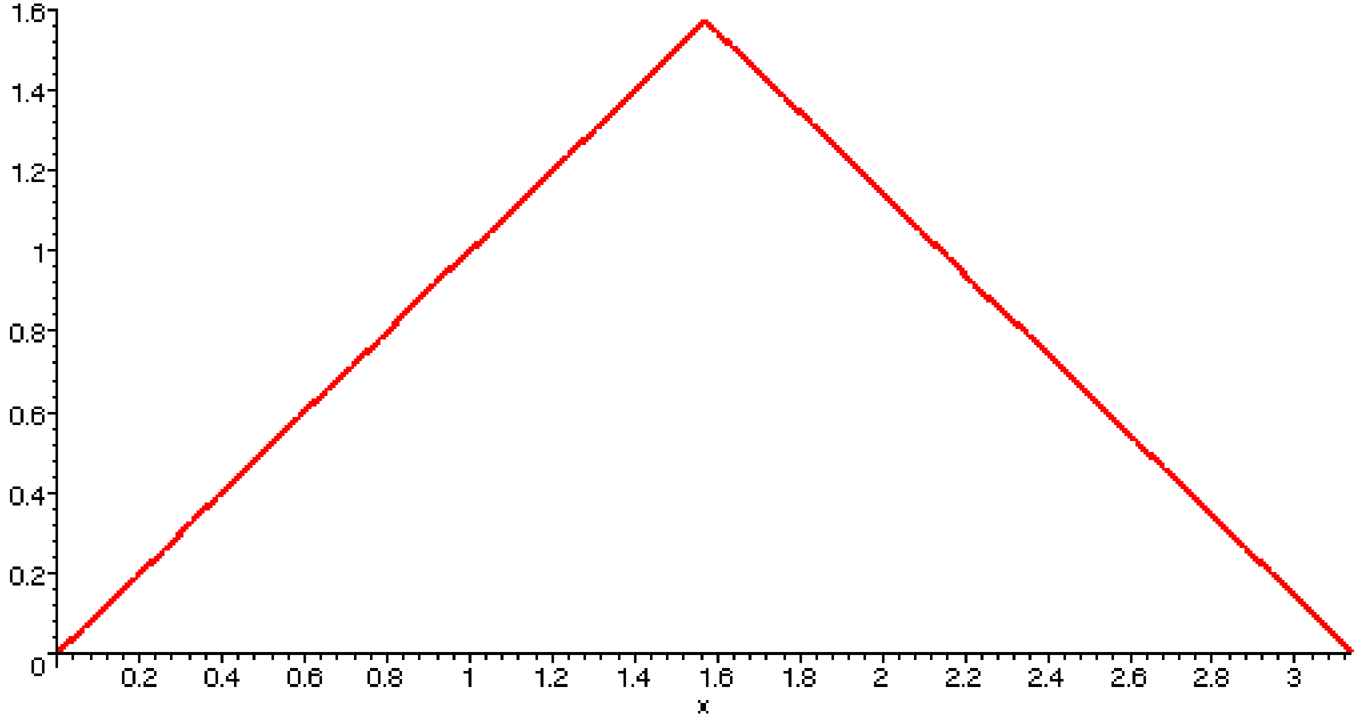
or sin(3 x) cos(3 t).

To see what happens to any other initial distribution, we make the Fourier series

$$f(x) = \sum_n a_n \sin(nx)$$

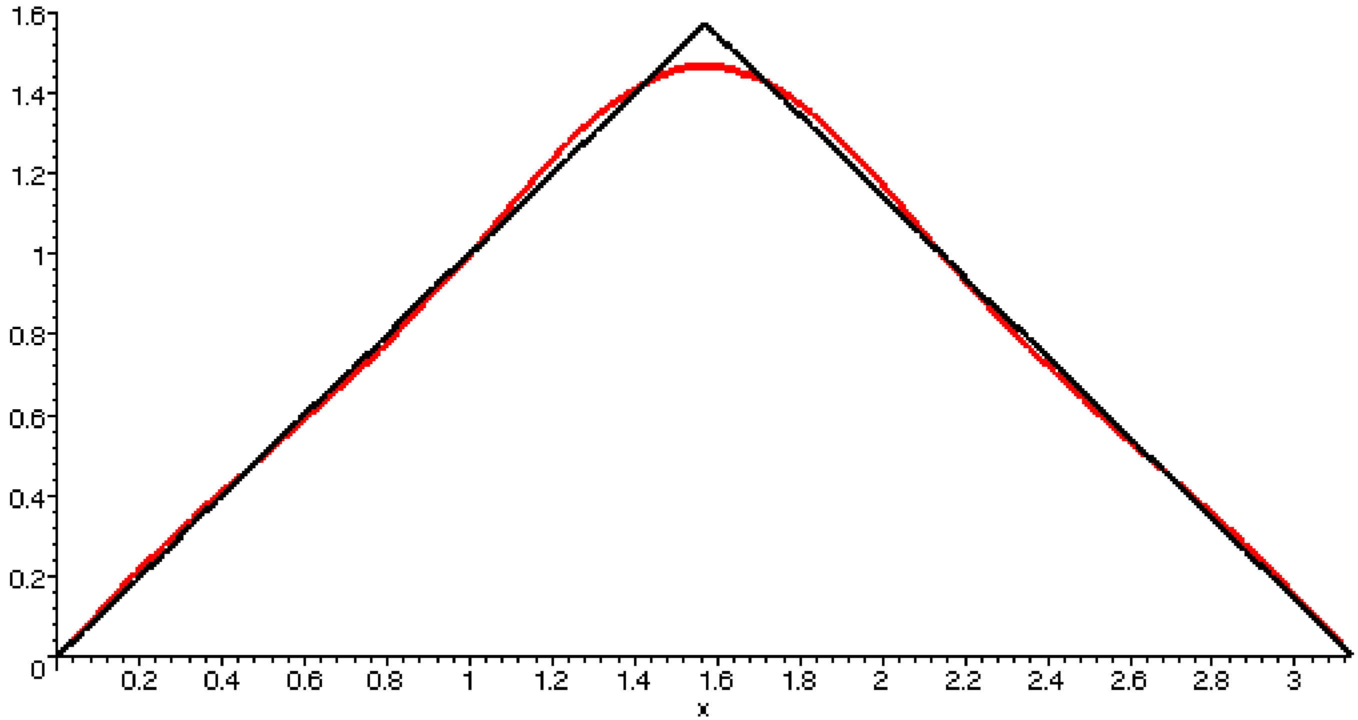
$$u(t, x) = \sum_n a_n \sin(nx) \cos(nt)$$

Graph of f:

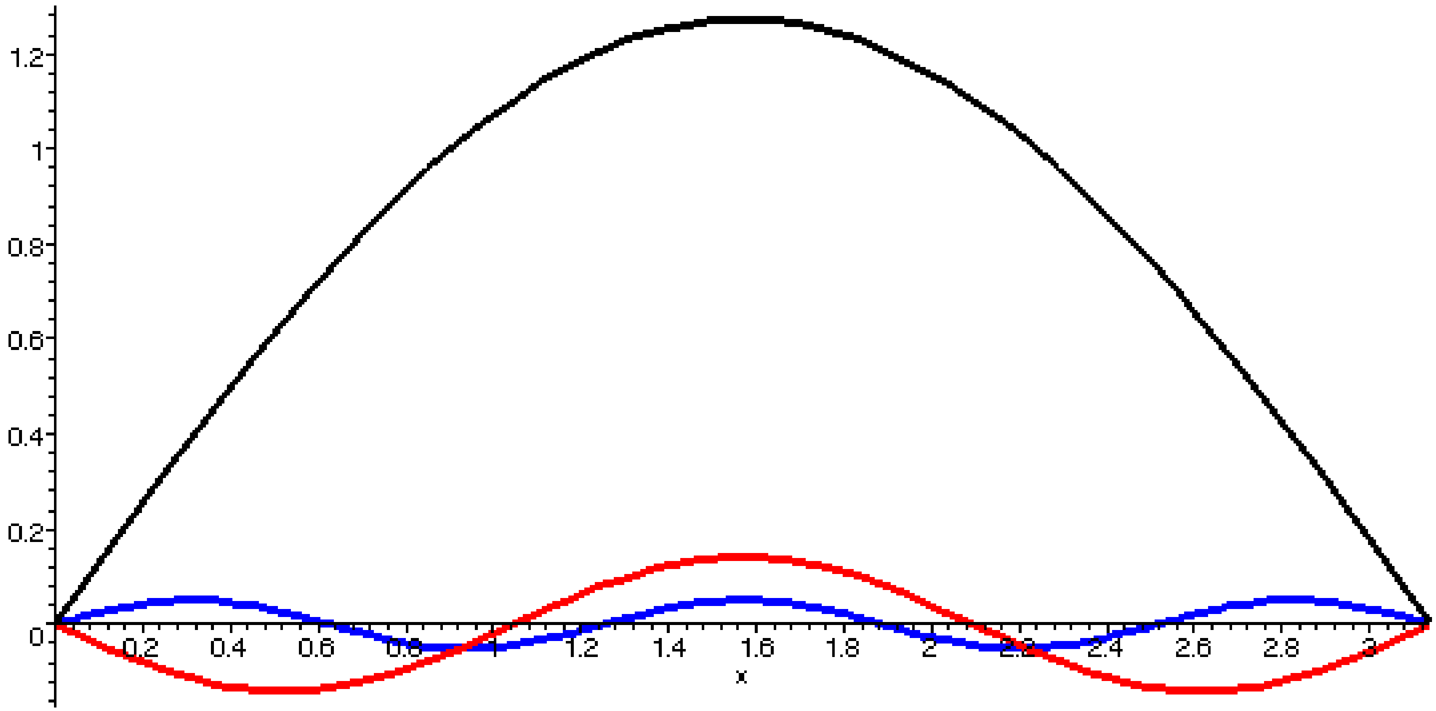


Approximation for f:

$f(x) = 4\sin(x)/\pi - 4\sin(3x)/9\pi + 4\sin(5x)/25\pi.$



Terms used:



What sounds good? You probably know that the C-major chord is C - E -G. Why it sounds good: mathematics and biology.

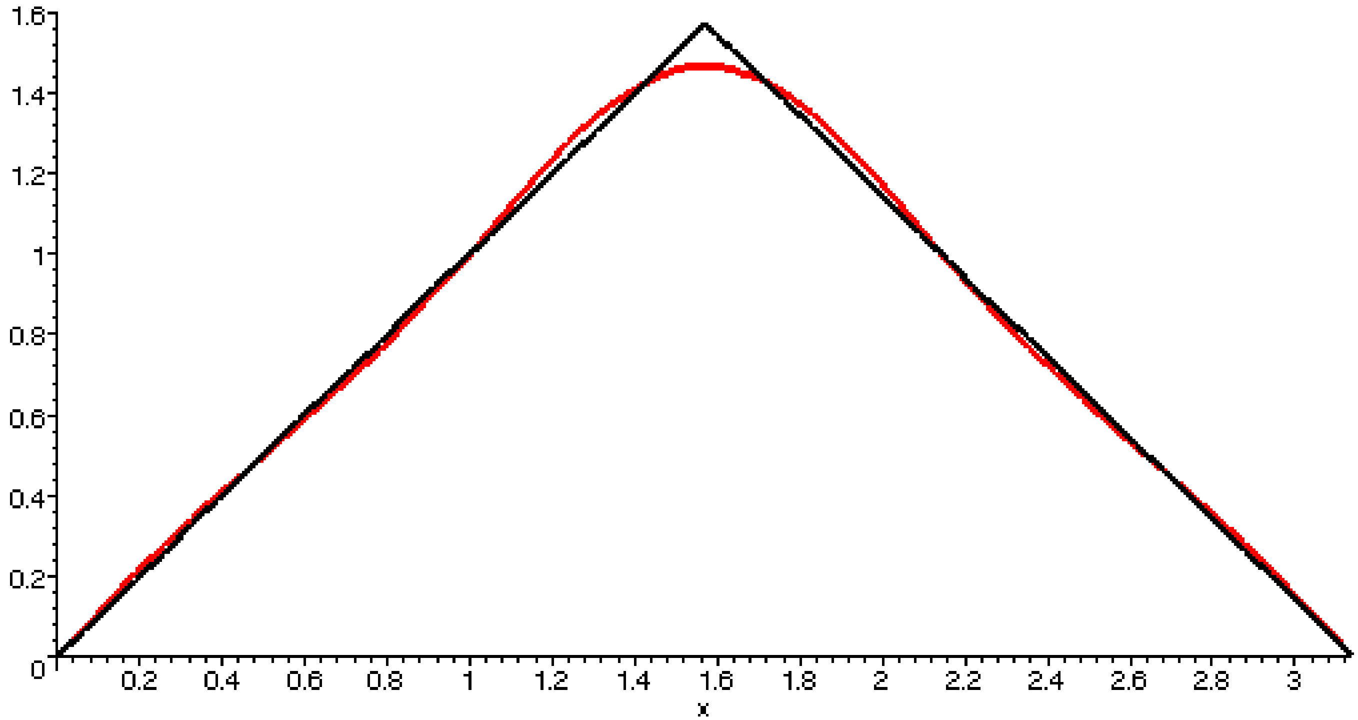
Plucked string could resemble C, together with faint overtones of E and G.

Remind you that if C = 256 vibrations per second, then E = ~323 and G = ~ 384.

256 3/2 = 384 and 256 5/4=320.

Approximation for f:

$$f(x) = 4\sin(x)/\pi - 4\sin(3x)/9\pi + 4\sin(5x)/25\pi.$$



This plucked string would be about C if the vibrations were 256 vibrations per second. There are those other terms. I have become deaf to higher pitched signals, especially as faint as the fourth, fifth, and higher order terms would be. Thus, I would hear only the first three terms anyway.

The mathematics of music would be an interesting direction to go from here. We won't.