Module 27 Laplace's Equation on a Rectangle

Partial Differential Equation: $u_t = u_{xx} + u_{yy}$

Boundary conditions:

$u(t,x,0) = f(x)$ and $u(t,x,L) = g(x)$

$u(t,0,y) = h(y)$ and $u(t,M,y) = J(y)$

Initial condition: $u(0,x,y) = K(x,y)$.

The steady-state problem first.

$$0 = u_{xx} + u_{yy}$$

Boundary conditions: $u(x,0) = f(x)$, $u(x,L) = g(x)$
$u(0,y) = h(y)$, $u(M,y) = J(y)$

And then the transition problem

$$u_t = u_{xx} + u_{yy}$$

Boundary conditions: $u(t,x,0) = 0$, $u(t,x,L) = 0$
$u(t,0,y) = 0$, $u(t,M,y) = 0$

Initial condition: $u(0,x,y) = k(x,y)$.

Solve the steady state problem as two problems.

For both, $0 = u_{xx} + u_{yy}$

Boundary conditions 1:

$u(x, 0) = f(x), \quad u(x, L) = g(x)$

$u(0, y) = 0, \quad u(M, y) = 0$

Boundary conditions 2:

$u(x, 0) = 0, \quad u(x, L) = 0$

$u(0, y) = h(y), \quad u(M, y) = J(y)$

Here are the steps:

(1) Make a separation of variables.

(2) Identify resulting ordinary differential equation and boundary conditions.

(3) Solve these equations.

(4) Construct a general solution for the problem.

(5) Use boundary conditions to get the solution.

Step 1: $X''Y + XY'' = 0$,

with $X(0)\,Y(y) = 0 = X(M)\,Y(y)$.

Step 2: $X'' + \lambda^2 X = 0$, $X(0) = 0 = X(M)$,

$Y'' = \lambda^2 Y$.

Step 3: $\lambda = n\pi/M$, $X(x) = \sin(n\pi x/M)$, and

$Y(y) = \sinh(n\pi y/M)$ and $\sinh(n\pi(L - y)/M)$.

Step 4: with $L = M = 1$,

$$U(x,y) = \sum_n (a_n \sinh(n\Pi y) + b_n \sinh(n\Pi(1 - y))\sin(n\Pi x)$$

Step 5: Use the boundary conditions to determine the a 's and b 's.

$$X''Y + XY'' = 0,\ X(0)Y(y) = 0 = X(M)Y(y)$$

leads to

$$U(x,y) = \sum_n (a_n \sinh(n\Pi y) + b_n \sinh(n\Pi(1-y))\sin(n\Pi x)$$

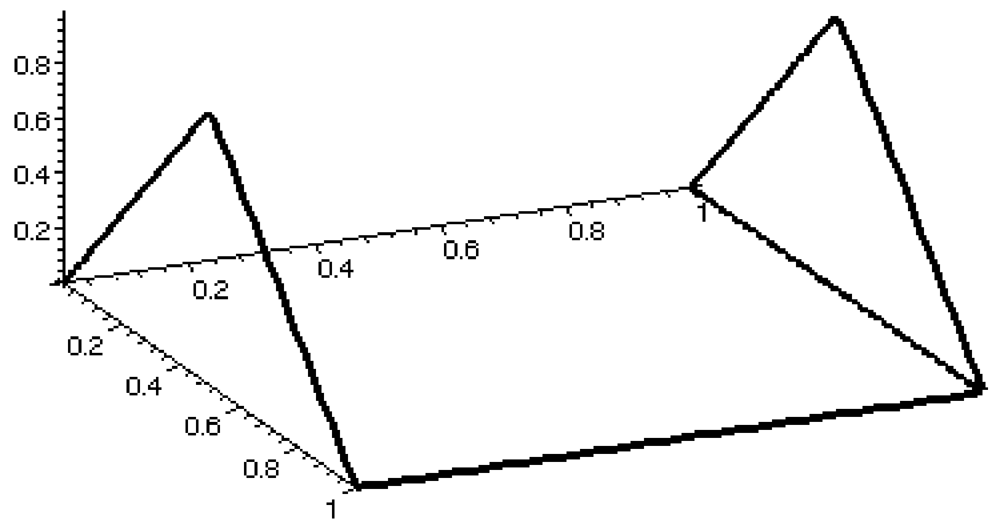
$$u(x, 0) = f(x), \quad u(x, 1) = g(x)$$

leads to

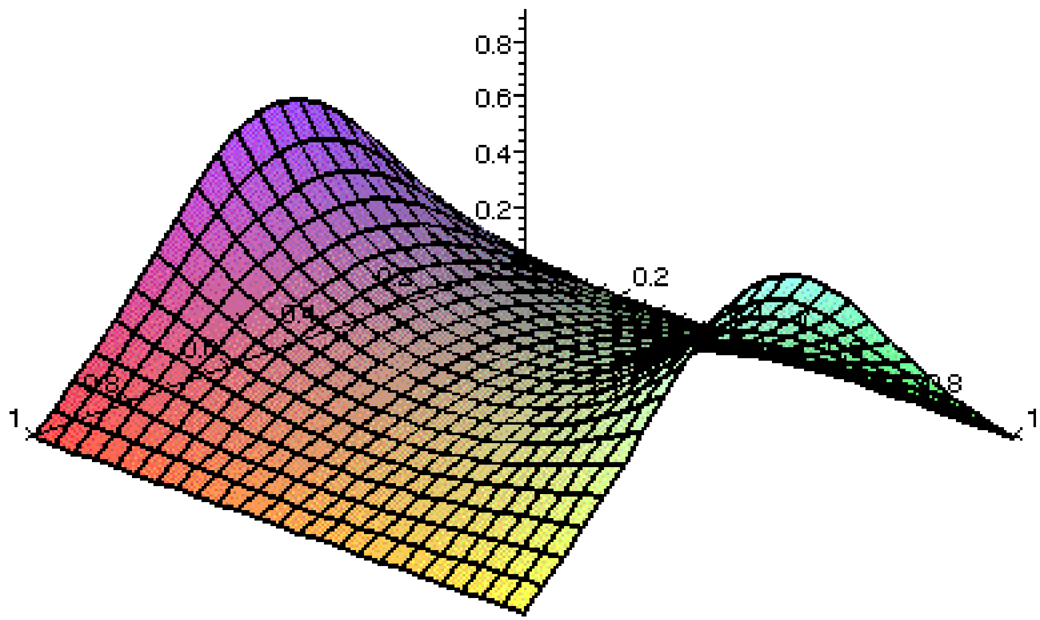
$$f(x) = \sum_n b_n \sinh(n\Pi)\sin(n\Pi x) \quad \text{and}$$

$$g(x) = \sum_n a_n \sinh(n\Pi)\sin(n\Pi x)\ .$$

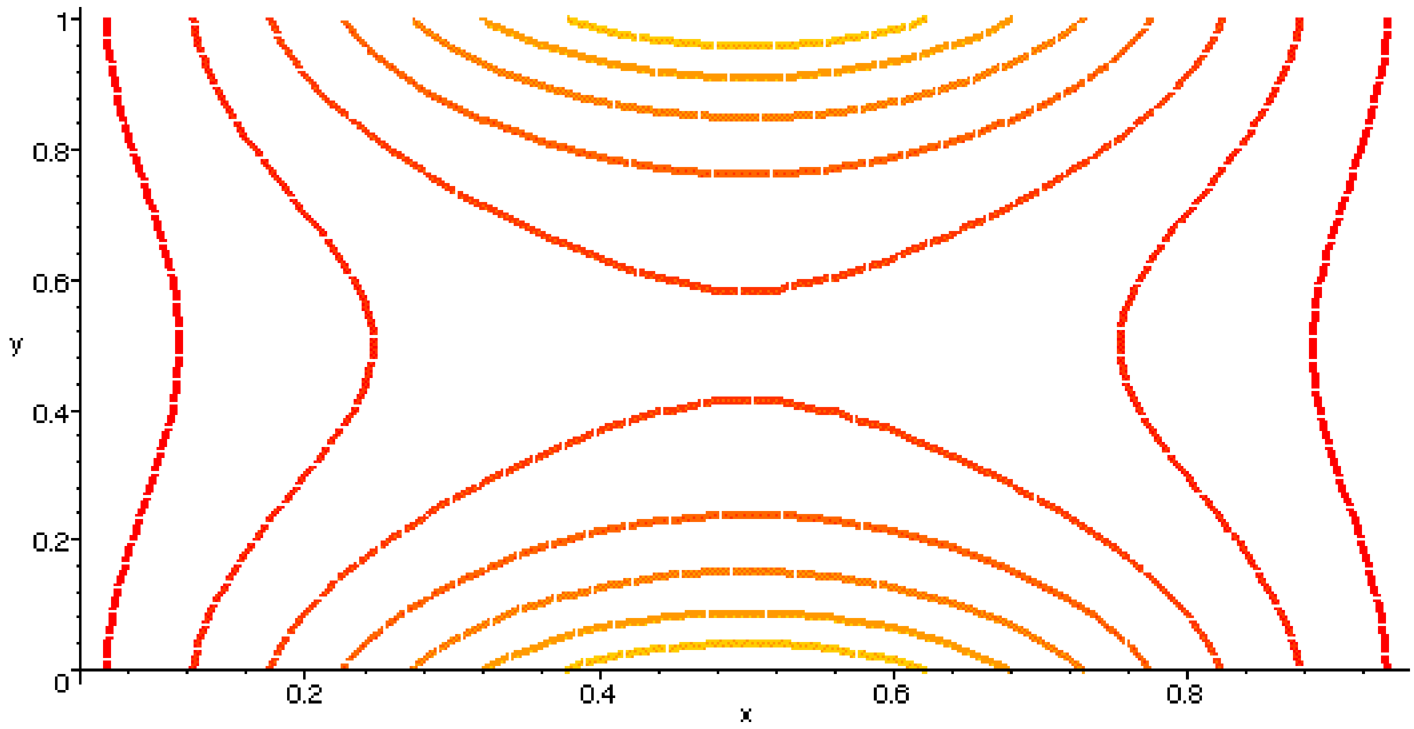
Setup with f(x) = g(x).



Graph of u(x, y)



Contour Plot



We now do a problem with two sides insulated.

For both, $0 = u_{xx} + u_{yy}$

Boundary conditions:

$u(x, 0) = f(x), \quad u(x, L) = f(x)$

$u_x(0, y) = 0, \quad u_x(M, y) = 0$

No passage across the left or right boundary of the rectangle, but the top and bottom of the distribution are held as f(x).

$X''Y + XY'' = 0$, $X'(0)\,Y(y) = 0 = X'(M)\,Y(y)$
leads to, where $L = M = 1$:

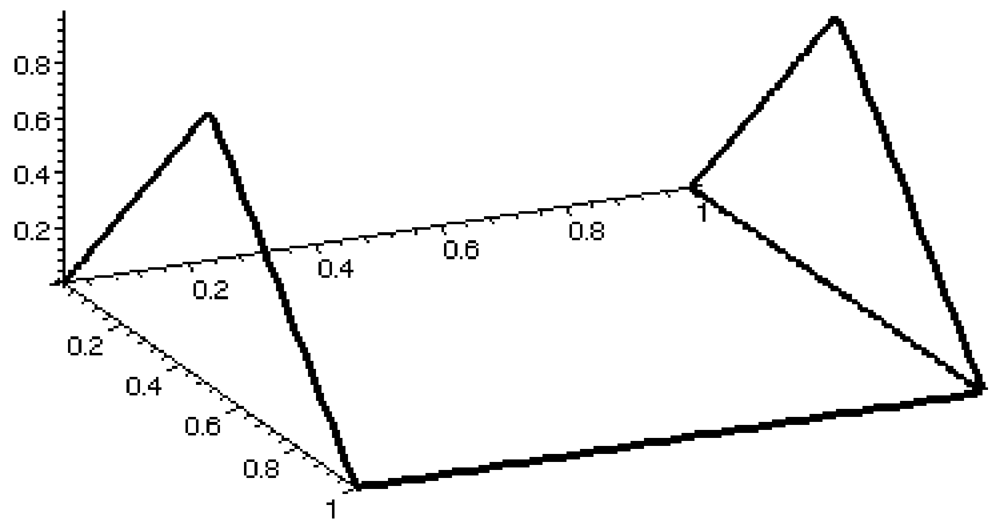
$U(x,y) = a_0 + (b_0 - a_0)\,y +$

$$\sum_n \left(a_n \sinh(n\Pi y) + b_n \sinh(n\Pi(1-y))\right)\cos(n\Pi x)$$

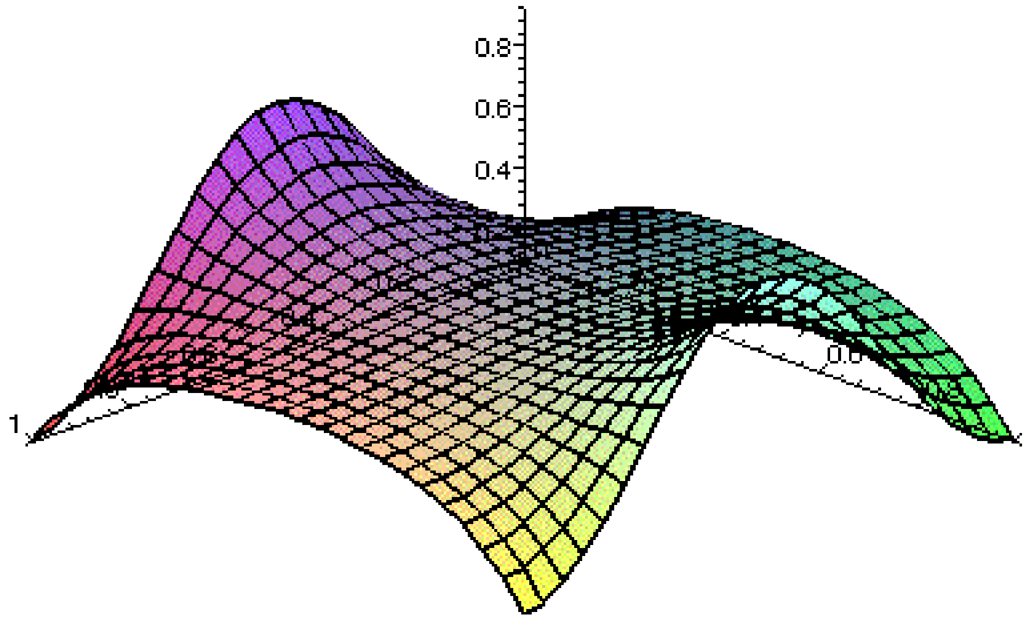
$u(x, 0) = f(x)$, $u(x, 1) = f(x)$
leads to

$f(x) = a_0 + \sum_n b_n \sinh(n\Pi)\cos(n\Pi x)$ and
$f(x) = b_0 + \sum_n a_n \sinh(n\Pi)\cos(n\Pi x)$

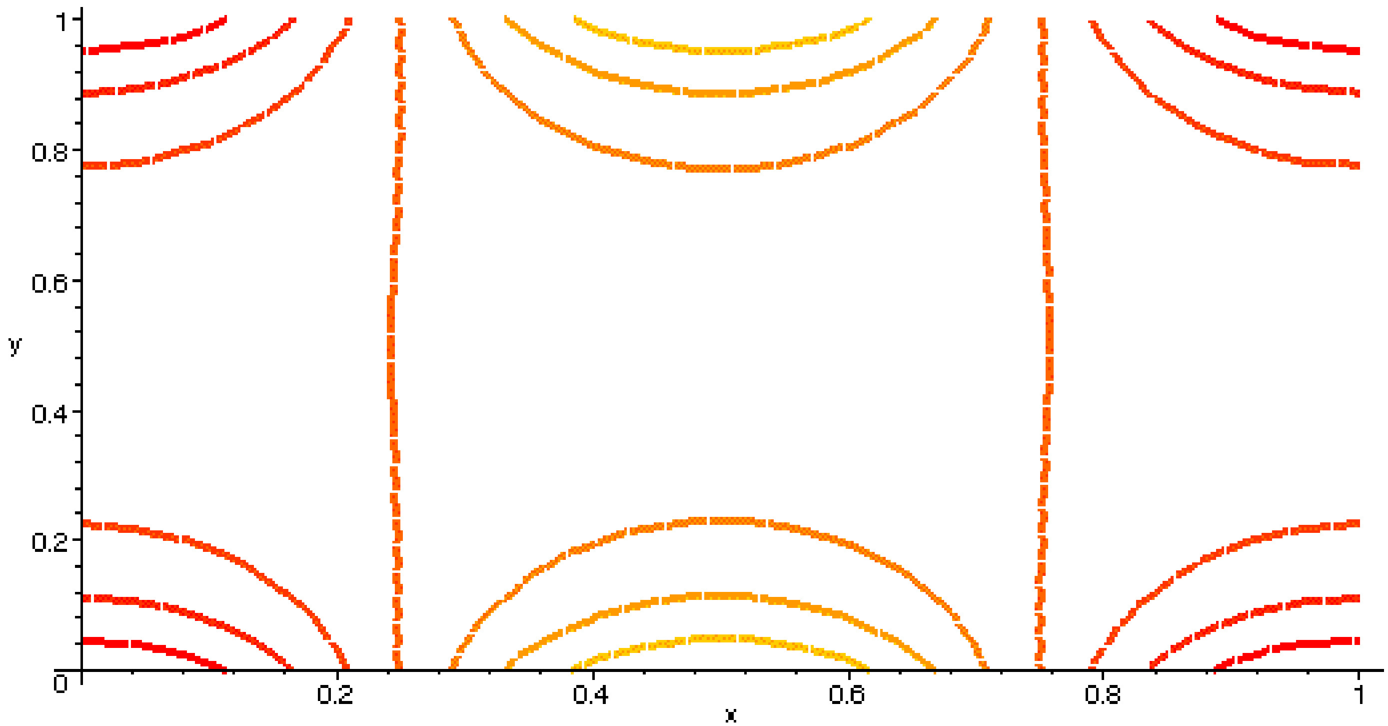
Setup with f(x) = g(x).



Graph of u(x, y)



Contour Plot



When values of u specified on the boundary the boundary conditions are called Dirchlet boundary conditions.

When values of the derivative of u are specified on the boundary, the boundary conditions are called Neumann boundary conditions.

Boundary conditions:

$$u(x, 0) = f(x), \quad u(x, L) = f(x)$$

$$u_x(0, y) = 0, \quad u_x(M, y) = 0$$

Assignment: See the Maple worksheet.

In this Module 27, we have

(1) introduced Laplace's equation on a rectangle,

(2) worked one problem with values specified on the boundary, and

(3) worked one problem with mixed boundary conditions.