Module 28: Laplace's Equation with Insulated Boundaries

We work the following problem

$$0 = u_{xx} + u_{yy}$$

Boundary conditions:

$$u_y(x,0) = 0, \qquad u_y(x,1) = 0$$

$$u(0,y) = \sin(\pi y), \qquad u_x(1,y) = 0$$

Not two PDE's for this classroom problem.

$$Y'' + \lambda^2 Y = 0,\ Y'(0) = 0 = Y'(1),$$

$$X'' - \lambda^2 X = 0.$$

Solutions:

$\lambda_n = n\pi.$

$X_0 = A_0 x + B_0$, $X_n = \sinh(n\pi x)$ or $\sinh(n\pi(1-x))$,

$Y_n(y) = \cos(n\pi y).$

$U(x,y) = A_0 x + B_0 +$

$$\sum_n (A_n \sinh(n\Pi x) + B_n \sinh(n\Pi(1-x))\cos(n\Pi y)$$

$$\sin(\pi y) = U(0, y) = B_0 + \sum_n B_n \sinh(n\Pi)\cos(n\Pi y).$$

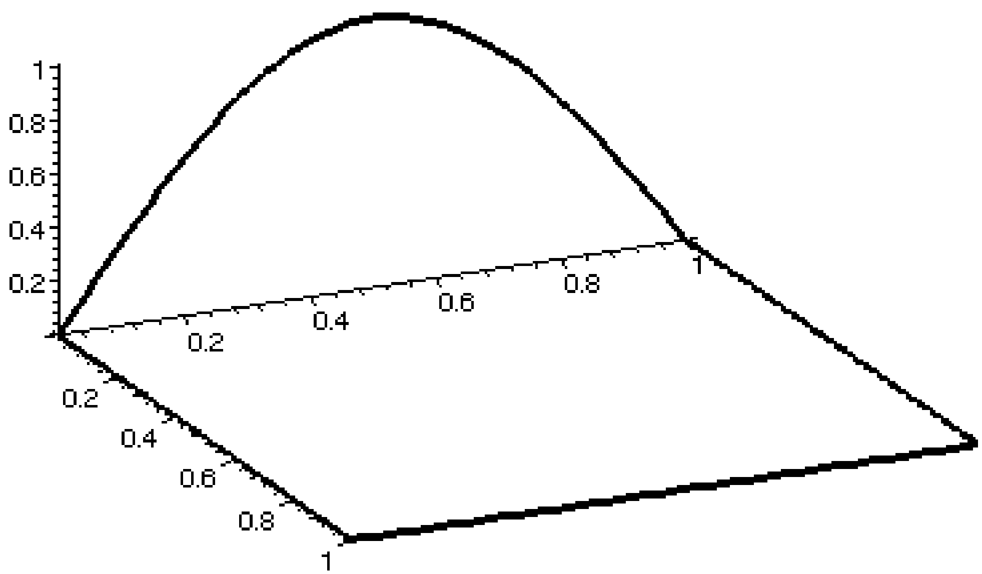
$$B_0 = \int_0^1 \sin(\Pi y)\,dy$$

$$B_n \sinh(n\pi) = 2\int_0^1 \sin(\Pi y)\cos(n\Pi y)\,dy$$

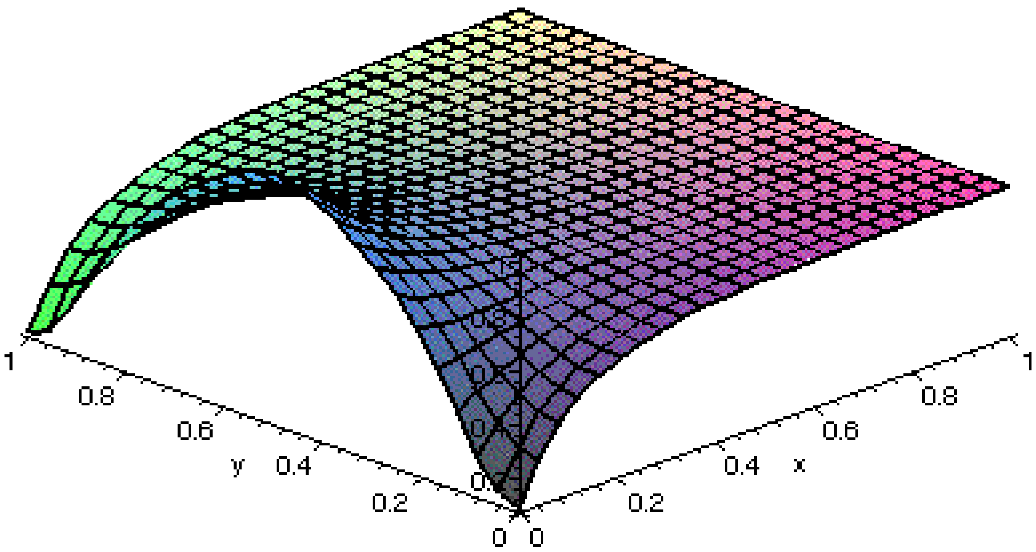
$$0 = u_x(1, y) = A_0 + \sum_n n\Pi(A_n \cosh(n\Pi) - B_n)\cos(n\Pi y)$$

$A_0=0$ and $A_n = B_n/\cosh(n\pi)$

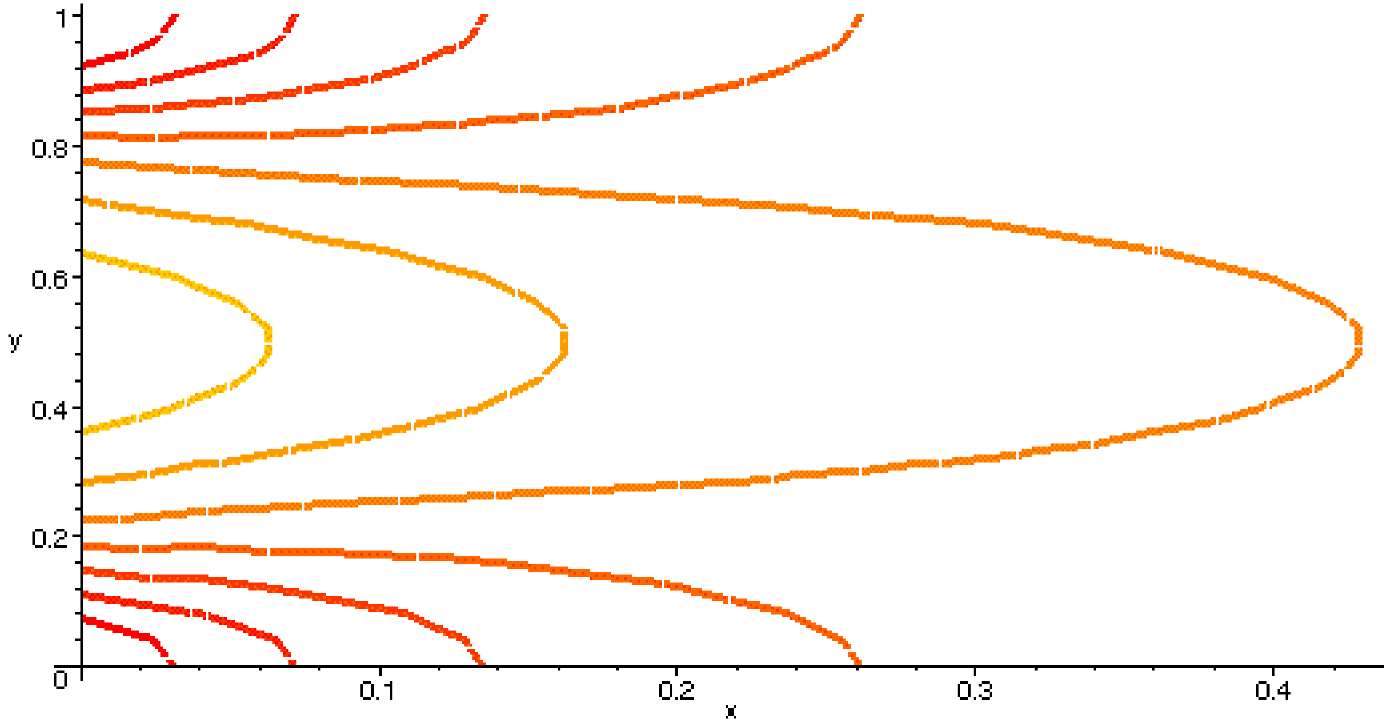
The situation:



Graph of u



Contour Lines



We work the following problem

$$0 = u_{xx} + u_{yy}$$

Boundary conditions:

$u_y(x,0) = 1 - \cos(2\pi x)$, $u_y(x,1) = 1$

$u_x(0,y) = 0$, $u_x(1,y) = 0$

That u is not changing in time has implications for what is mathematically possible here.

We know how to separate variables. The differential equations will be

$$X'' + \lambda^2 X = 0, \; X'(0) = 0 = X'(1),$$

and $$Y'' = \lambda^2 Y.$$

Solutions for this will be

$X(x) = 1$ or $\cos(n\pi x)$ if $n > 0$

and $Y(y) = 1$ and y

or $\cosh(n\pi y)$ and $\cosh(n\pi(1-y))$ if $n > 0$.

$X(x) = 1$ or $\cos(n\pi x)$ if $n > 0$

and $Y(y) = 1$ and y
or $\cosh(n\pi y)$ and $\cosh(n\pi(1-y))$ if $n > 0$.

We can construct a general solution.

$$U(x, y) = A_0 + B_0 y +$$

$$\sum_n [A_n \cosh(n\Pi y) + B_n \cosh(n\Pi(1-y))]\cos(n\Pi x)$$

We have only to find the A 's and B 's.

Coming in from the bottom is

$$f(x) = 1 - \cos(2\pi x) = u_y(x, 0)$$

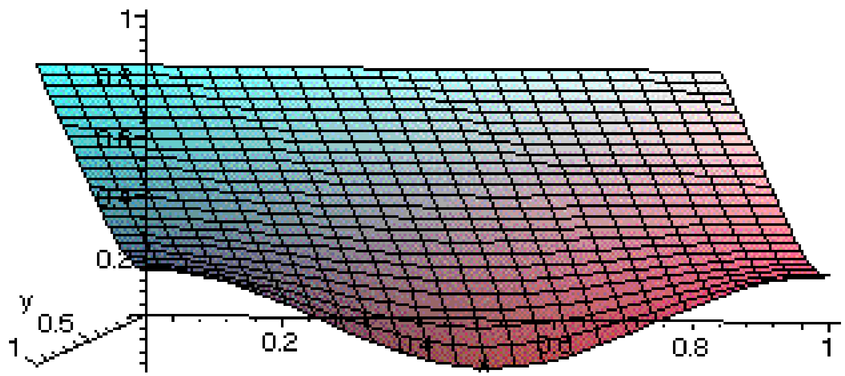
$$= B_0 - \sum_n B_n n\Pi \sinh(n\Pi)\cos(n\Pi x)$$

Going out the top is

$$1 = u_y(x, 1) = B_0 - \sum_n A_n \sinh(n\Pi)\cos(n\Pi x)$$

A job for Fourier Series?

Graph of u



Assignment: See the Maple worksheet

In this Module 28 we have worked problems which had Neumann boundary conditions.