Module 29: The Structure of Solutions

In this module, we discuss properties of solutions for Laplace's Equation.

The Maximum Principle for solutions of Laplace's Equation is a generalization of the following familiar idea from the calculus.

Theorem 1: Suppose that $0 \leq u''$ on [0,1]. Then the maximum value of u occurs at 0 or 1.

As is common, we use $\Delta(u)$ to represent the Laplacian Operator:

$$\Delta u = u_{xx} + u_{yy}.$$

It is common to use ∂D to represent the boundary of a region D.

For example,

$\Delta u = 0$ if $u(x, y) = \sin(3x) \sinh(3y)$,

∂D is a circle if D is a disk, and

∂D is the entire real line if D is the upper-half plane.

Theorem 2: Suppose that f is continuous on a closed and bounded rectangle D. Let u be continuous on D and

$$\Delta u = f \text{ on the interior of D.}$$

If $0 \le f(x, y)$, then u assumes its maximum value on $\partial(D)$.

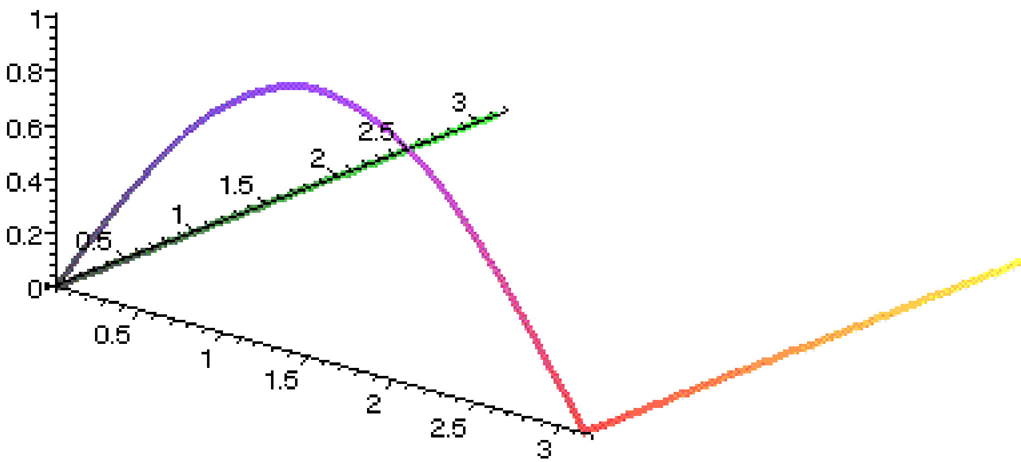
Indication of Proof: Suppose $0 < f$. Any function continuous on a closed and bounded set in R^n has a maximum on that set. Suppose the maximum is in the interior of D. Then the first partials of u are zero and the second partials are not positive. This is a contradiction to $\Delta u = f > 0$. It must be that the point at which the maximum occurs is not in the interior of D. The only choice is that the maximum occurs on the boundary of D. If we have only that $f \geq 0$, there is more to say.

Actually, this maximum principle holds for any reasonably nice, bounded region with boundary.

Example 1. Let u satisfy $\Delta u = 0$ on the unit disk with $u(1, \Theta) = \sin(\Theta)$. Then $u(r, \Theta) = r \sin(\Theta)$. It is clear that this u takes on its maximum value on the boundary of the unit disk -- on the unit circle.

Wait until the next module to see how to evaluate Δu in polar coordinates.

Example 2. Let D be the half infinite strip with $0 \leq x \leq \pi$, $y > 0$, and $g(x,y) = \sin(x)$ on the boundary. Let $u(x,y) = \sin(x) \exp(y)$.



The maximum principle implies the following. This result is important for scientists and engineers.

Theorem 3: (Continuous Dependence on the Data) Let D and u be as in Theorem 2. Let g be u on the boundary of D and $f = \Delta u$ on the interior of D. There is a number k such that

$$| u | \leq \max | g | + k \max | f |.$$

Theorem 4: Suppose D is as in Theorem 2 and both u and v satisfy the same PDE and boundary conditions. Then u = v.

Indication of proof: Use the Theorem 3, which said:

$$| u | \leq \max | g | + k \max | f |.$$

The "Behold! Method" for making solutions:

Solve $\Delta u = 0$,

With $u(x, 0) = 0$, $u(x, \pi) = 5 \sin(7 x)$,
And $u(0, y) = 0$, $u(\pi, y) = 0$.

Behold!: $5 \sin(7 x) \sinh(7 y)/\sinh(7 \pi)$.

Theorem 4: Suppose that u is a solution for the problem

$$\Delta u = 0,$$

with $u_y(x, 0) = f(x)$, and $u_y(x, M) = g(x)$,

$u_x(0, y) = h(y)$ and $u_x(L, y) = j(y)$.

Then,

$$\int_0^L g(x)dx - \int_0^L f(x)dx + \int_0^L j(y)dy - \int_0^L h(y)dy = 0.$$

Assignment: See the Maple worksheet.

In this Module 29, we have presented

(1) The maximum principle for solutions of a Dirchlet Problem,

(2) The continuous dependence on data,

(3) The uniqueness result, and

(4) The compatibility condition for Neumann Problems.