Module 30: Laplace's Equation on a Disk

The form for Laplace's Equation, $\Delta u = 0$, depends on which coordinate system is being used. With rectangular coordinates,

$$\Delta u = u_{xx} + u_{yy}.$$

It would be no surprise how to write this in three dimensions, instead of two.

What about a disk,

or annulus,

or cylinder?

Think polar coordinates.

$x = r\cos(\theta)$ and $y = r\sin(\theta)$.

Or, $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(y / x)$

For example take these situations

[1, 1]	$\leftrightarrow$	$r = \sqrt{2}$	$\theta = \pi/4$,
[1, -1]	$\leftrightarrow$	$r = \sqrt{2}$	$\theta = -\pi/4$,
[-1, 1]	$\leftrightarrow$	$r = \sqrt{2}$	$\theta = 3\pi/4$,
[-1, -1]	$\leftrightarrow$	$r = \sqrt{2}$	$\theta = -3\pi/4$,

Laplace's Equation in Polar Coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \Theta^2} = 0.$$

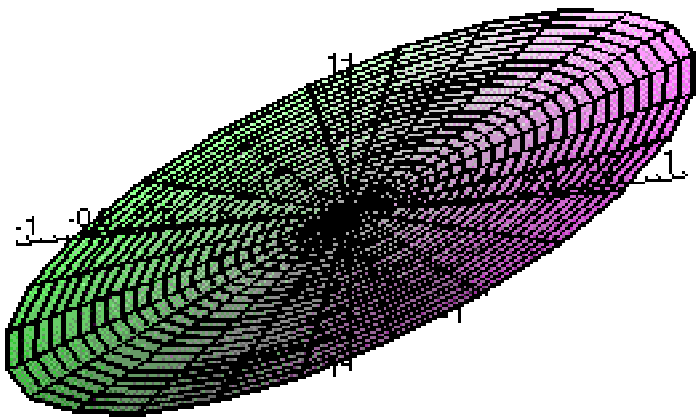
A typical problem on a disk:

$$\Delta u = 0 \text{ for } 0 < r < 1, -\pi < \theta < \pi$$

Warning: Hidden Conditions.

With u specified at $r = 1$, solve if $u(1, \theta) = f(\theta)$.

Guess the solution if $u(1, \theta) = \cos(\theta)$.



Separation of Variables for Laplace's Equation in Polar Coordinates.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \Theta^2} = 0$$

becomes

$$1/r \; (r R')' \, \theta + 1/r^2 \; R \, \theta'' = 0$$

$$r^2 R'' + r R' - \mu R = 0,$$

$$\theta'' + \mu \theta = 0.$$

$$r^2 R'' + r R' - \mu R = 0,$$

$$\Theta'' + \mu \Theta = 0.$$

$$\Theta(-\pi) = \Theta(\pi), \qquad \Theta'(-\pi) = \Theta'(\pi),$$

Familiar territory: $\mu = -n^2$

$\Theta(\theta) = 1$ or $\sin(n\theta)$ and $\cos(n\theta)$.

$R(r) = r^n$ or r^{-n}.

There is reason to get

$$R(r) = r^n.$$

Thus

$$U(r, \theta) = a_0 + \sum_n a_n r^n \cos(n\theta) + \sum_n b_n r^n \sin(n\theta)$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt \quad \text{and} \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt$$

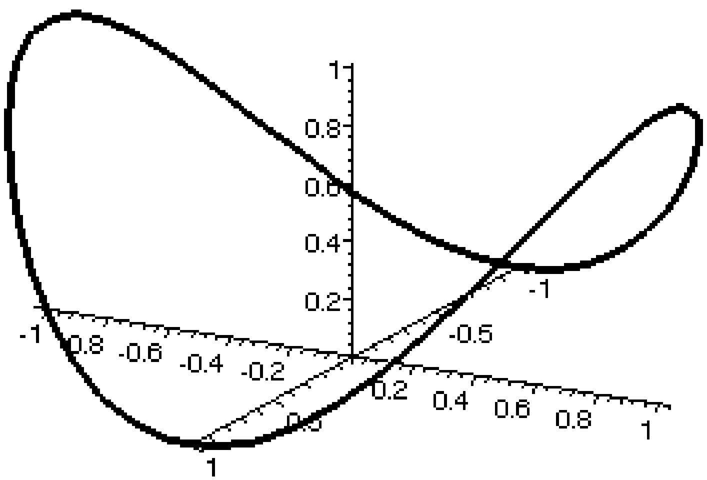
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt$$

Let's do an example more complicated than

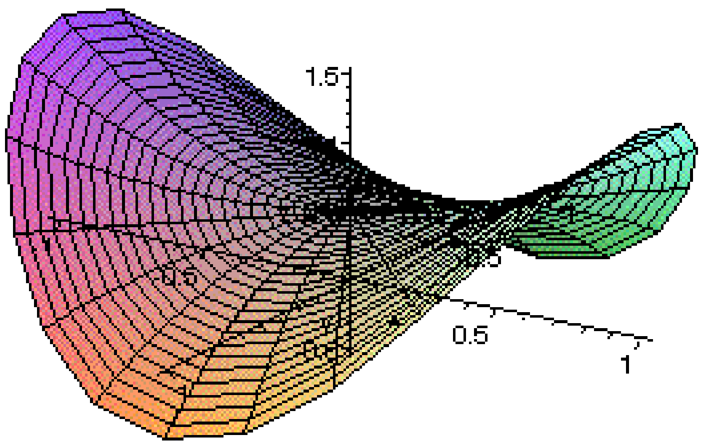
$u(1, \theta) = \cos(\theta)$.

Say, $u(1, \theta) = \sin(\theta)^2$

Here is a picture of the boundary condition.



Here is how to think of the solution.



Summary for a disk with radius c:

$$u(r, \theta) = a_0 + \sum_n a_n r^n \cos(n \theta) + \sum_n b_n r^n \sin(n \theta).$$

$$u(r, \theta) = 1/\pi \int_{-\pi}^{\pi} [1/2 + \sum_n (r/c)^n \cos(n (t-\theta))] f(t)\, dt$$

$$= 1/2\pi \int_{-\pi}^{\pi} P(r, t, \theta) f(t)\, dt .$$

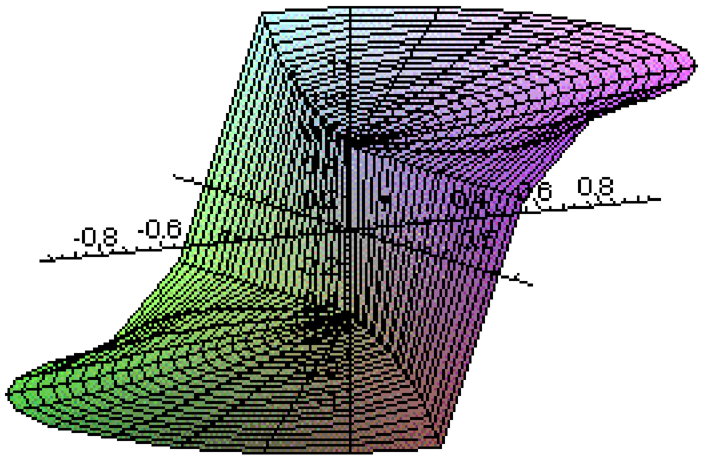
This formula for u does not appear to depend on Fourier Series. The job of computing solutions has now been reduced to the job of computing one integral. Maybe this makes the problem easy, maybe that integral can be hard.

Here is an example: Take $f(\theta) = 1$ on $[-\pi/2 , \pi 2]$.

We compute $u(r, \theta) =$

$$1/2\pi \int_{-\Pi/2}^{\Pi/2} (1 - r^2) / (1 + r^2 - 2 r \cos(t - \theta)) \, 1 \, dt$$

Here is the solution:



Assignment: See the Maple Worksheet.

In this Module 30 we have changed Laplace's Equation to polar coordinates and solved three problems -- one by the "Behold!" method, one by separation of variables, and one by using Poisson's Integral Formula.