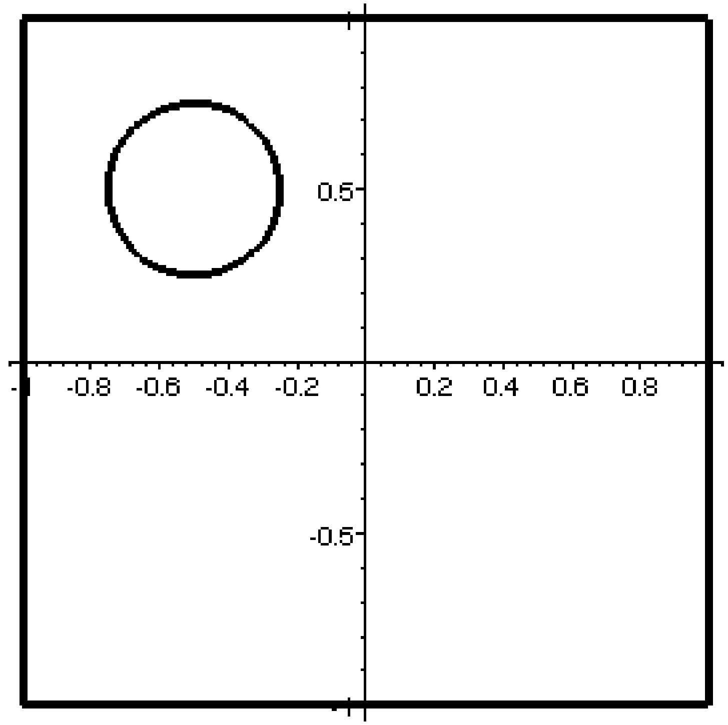
Module 31: Laplace's Equation on a Ring or a Half Disk

Suppose that u is the solution for

$$\Delta u = 0$$

on any region. Take p_0 to be any point on the interior of the region. Take C to be a circle about p_0 so the circle lies in the region. We want to show that $u(p_0)$ is the average value of u over C.



$$u(p_0) = \int u(x(t),y(t))dt \Big/ 2\Pi$$

Here's a way to think about this:

$$u(r, \theta) = a_0 + \Sigma_n a_n r^n \cos(n\theta) + \Sigma_n b_n r^n \sin(n\theta)$$

realized for this new disk in the region.

$$u(p_0) = u(0, \theta) = a_0 = \int u(x(t),y(t))dt \Big/ 2\Pi \; .$$

What is the significance of this result?

It would imply that the value of u at p_0 is the average of the values of u on any circle about p_0 which lies in the region on which u is defined. This gives an understanding of the Maximum Principle.

How could u have a maximum at p_0 if it is the average of values all around it?

The idea generalizes to three dimensions, or more.

Think of a cube with specified, unchanging temperature, perhaps different on each face. The temperature at each point on the interior will be exactly the average of the surrounding temperatures. This seems to be an interesting way to conceive the construction of the heat distribution. One might think it would not be possible to make such a distribution knowing only the temperatures on the boundaries.

That it is possible is surely a tribute to our predecessors: Fourier, Cauchy, Laplace, Poisson, etc.

Laplace's Equation on a Ring

We find a function u which satisfies

$$\Delta u = 0, \quad \text{for } 1 \le r \le 2, \; -\pi < \theta \le \pi$$

with

$$u(1, \theta) = u(2, \theta) = \sin(\theta) \text{ for } -\pi < \theta \le \pi.$$

Remember the form:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \Theta^2} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\left(\frac{\partial}{\partial r}v\right)\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}v = 0$$

becomes

$$r^2 R'' + r R - \mu R = 0,$$

$$\Theta'' + \mu \Theta = 0,$$

$$\Theta(-\pi) = \Theta(\pi), \quad \Theta'(-\pi) = \Theta'(\pi),$$

leading to

$$n=0,\ \Theta(\theta) = 1,\ R(r) = 1 \text{ and } \ln(r)$$

or

$$n > 0,\ \Theta(\theta) = \sin(n\,\theta) \text{ and } \cos(n\,\theta)$$

$$u(r,\theta) = a_0 + \Sigma_n\, a_n\, r^n \cos(n\,\theta) + \Sigma_n\, a_{-n}\, r^{(-n)} \cos(n\,\theta)$$

$$+\, b_0 \ln(r) + \Sigma_n\, b_n\, r^n \sin(n\,\theta) + \Sigma_n\, b_{-n}\, r^{(-n)} \sin(n\,\theta)$$

$$\sin(\theta) = u(1,\text{theta}) = a_0 + \Sigma_n\, a_n \cos(n\,\theta) + \Sigma_n\, a_{-n} \cos(n\,\theta)$$

$$+\, \Sigma_n\, b_n \sin(n\,\theta) + \Sigma_n\, b_{-n} \sin(n\,\theta)$$

$$\sin(\theta) = u(2,\text{theta}) =$$

$$a_0 + \Sigma_n\, a_n\, 2^n \cos(n\,\theta) + \Sigma_n\, a_{-n}\, 2^{(-n)} \cos(n\,\theta)$$

$$+\, b_0 \ln(2) + \Sigma_n\, b_n\, 2^n \sin(n\,\theta) + \Sigma_n\, b_{-n}\, 2^{(-n)} \sin(n\,\theta)$$

Conclusion = ?

Conclusion:

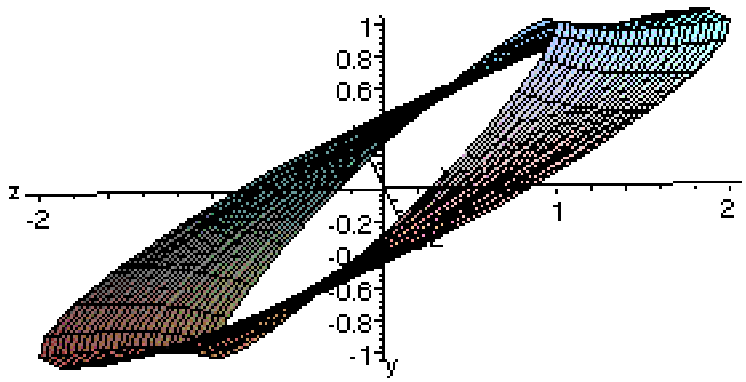
$b_1 + b_{-1} = 1$ and $2\, b_1 + b_{-1}/2 = 1$,

so that $b_1 = 1/3$ and $b_{-1} = 2/3$.

$u(r, \theta) = (r + 2/r) \sin(\theta) /3$.

Check it: $u(1, \theta) = \sin(\theta) = u(2, \theta)$.

Graph of $u(r, \theta) = (r + 2/r)\sin(\theta)/3$.



Laplace's Equation on a Half Disk

We suppose that u satisfies

$$\Delta u = 0 \text{ for } 0 < r < 1,$$

with $u(r, 0) = 0 = u(r, \pi)$, $u(r, 1) = 1$.

This leads to the ordinary differential equations with boundary conditions

$$r^2 R'' + r R - \mu R = 0,$$

$$\theta'' + \mu \theta = 0.$$

$$\Theta(0) = 0 \text{ and } \Theta(\pi) = 0$$

$$\Theta'' + \mu \Theta = 0.$$

$$\Theta(0) = 0 \text{ and } \Theta(\pi) = 0$$

It follows that $\lambda_n = n$,

$$\Theta(\theta) = \sin(n\theta),$$

and $$R(r) = r^n.$$

General solution is

$$u(r, \theta) = \Sigma\, a_n\, r^n \sin(n\theta)$$

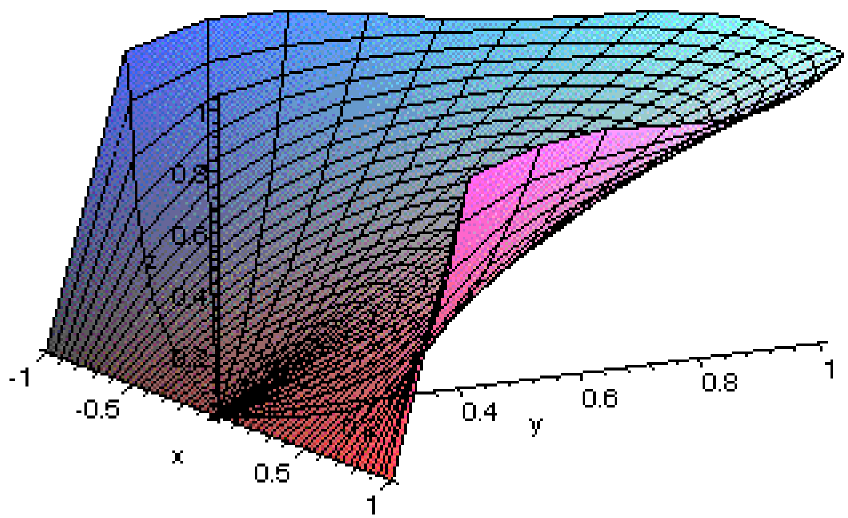
The requirement is that

$1 = u(1, \theta) = \Sigma\, a_n\, 1^n \sin(n\,\theta).$

Fourier sine series for the function 1 on $(0, \pi)$. Each term of the sum will be zero at zero and at π. Expect the convergence to be NOT uniform.

$$a_n = \int_0^{\Pi} \sin(n\Theta)\, d\Theta \Big/ \int_0^{\Pi} \sin(n\Theta)^2\, d\Theta$$

Graph of solution:



Assignment: See Maple worksheet

In this Module 31, we have

(1) discussed the average value property,

(2) found solutions for Laplace's Equation on a ring, and

(3) found solutions for Laplace's Equation on a half disk.