Module 32: Heat Equation on a Rectangle

We consider the diffusion of heat into a long beam with cross section a rectangle. The supposition that the beam is "long" is to produce the mathematical idea that heat diffusing to a point comes essentially only from the sides and that the ends are so far away that heat coming from above or below can be ignored. This is the same problem as considering heat diffusion in a thin rectangular plate that is insulated at the top and bottom.

The problem leads to an equation such as

$$u_t(t, x, y) = u_{xx} + u_{yy}$$

with boundary conditions

$$u(t, x, 0) = f_0, \quad u(t, x, b) = f_1,$$
$$u(t, 0, y) = g_0, \quad u(t, a, 0) = g_1,$$

and initial conditions

$$u(0, x, y) = h(x, y).$$

The problem is broken into two parts: the steady state v(x,y) and the transient, w(t, x, y).

The steady-state problem is

$$0 = v_{xx} + v_{yy}$$

with boundary conditions

$$v(x, 0) = f_0, \quad v(x, b) = f_1,$$
$$v(0, y) = g_0, \quad v(a, 0) = g_1.$$

We have discussed how to break this problem into two parts previously. They would be

$$0 = v_{1\,xx} + v_{1\,yy}$$

with boundary conditions

$$v_1(x,0) = f_0,\ \ v_1(x,b) = f_1,$$
$$v_1(0,y) = 0,\ \ v_1(a,y) = 0,$$

and

$$0 = v_{2\,xx} + v_{2\,yy}$$

with boundary conditions

$$v_2(x,0) = 0,\ \ v_2(x,b) = 0,$$
$$v_2(0,y) = g_0,\ \ v_2(a,y) = g_1.$$

Steady state solution

$$v(x, y) = v_1(x, y) + v_2(x, y).$$

Make the computation for the transient solution.

Transient problem is

$$w_t(t, x, y) = w_{xx} + w_{yy}$$

with boundary conditions

$$w(t, x, 0) = 0, \quad w(t, x, b) = 0$$

$$w(t, 0, y) = 0, \quad w(t, a, y) = 0$$

with initial condition

$$w(0, x, y) = h(x, y) - v(x, y).$$

Consider: $u(t, x, y) = w(t, x, y) + v(x, y)$

We break equation

$$w_t(t, x, y) = w_{xx} + w_{yy}$$

into three ODE's, two of which have boundary conditions.

$X'' = -\lambda^2 X$, $\quad Y'' = -\mu^2 Y \quad T' = -(\lambda^2 + \mu^2)\, T$

$X(0) = X(a) = 0. \quad Y(0) = Y(b) = 0.$

$X'' = -\lambda^2 X$, $\quad Y'' = -\mu^2 Y \quad T' = -(\lambda^2 + \mu^2)\, T$

$X(0) = X(a) = 0. \quad Y(0) = Y(b) = 0.$

General solution of the transient equation:

$w(t, x, y) =$

$$\Sigma_n \Sigma_m A_{nm} \sin(n\pi x/a) \sin(m\pi y/b) \exp(-((n/a)^2+(m/b)^2)\pi^2 t)$$

The coefficients are obtained as

$$A_{nm} = 4\iint w(0,x,y)\sin(n/ax\pi)\sin(m/by\pi)dxdy/ab$$

Example

$$u_t(t, x, y) = u_{xx} + u_{yy}$$

Take boundary conditions as follows:

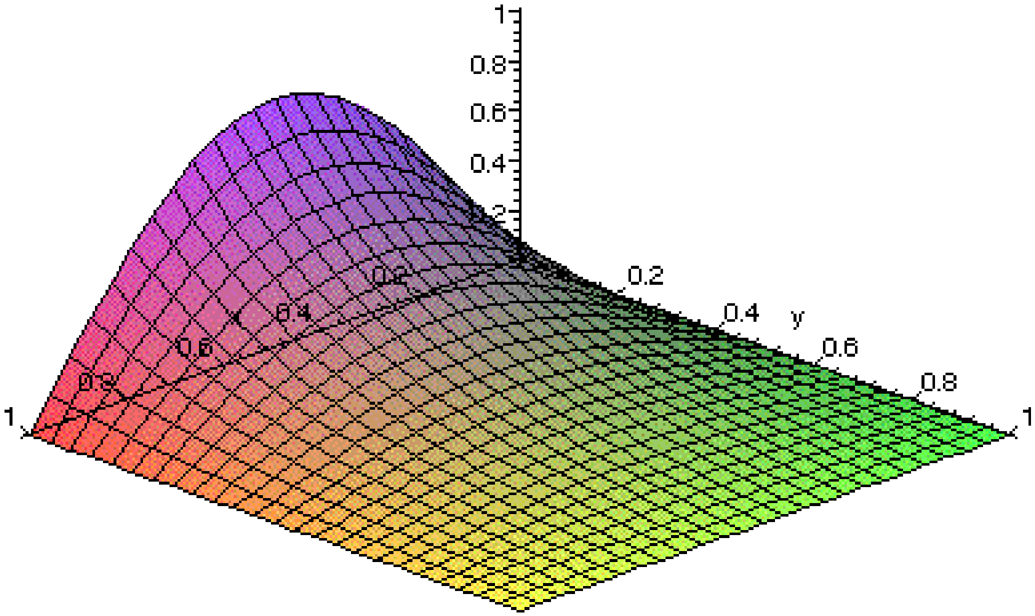
$$u(t, x, 0) = \sin(\pi x), \quad u(t, x, 1) = 0$$
$$u(t, 0, y) = 0, \qquad u(t, 1, y) = 0$$

Take the initial condition as

$$u(0, x, y) = \sin(\pi x).$$

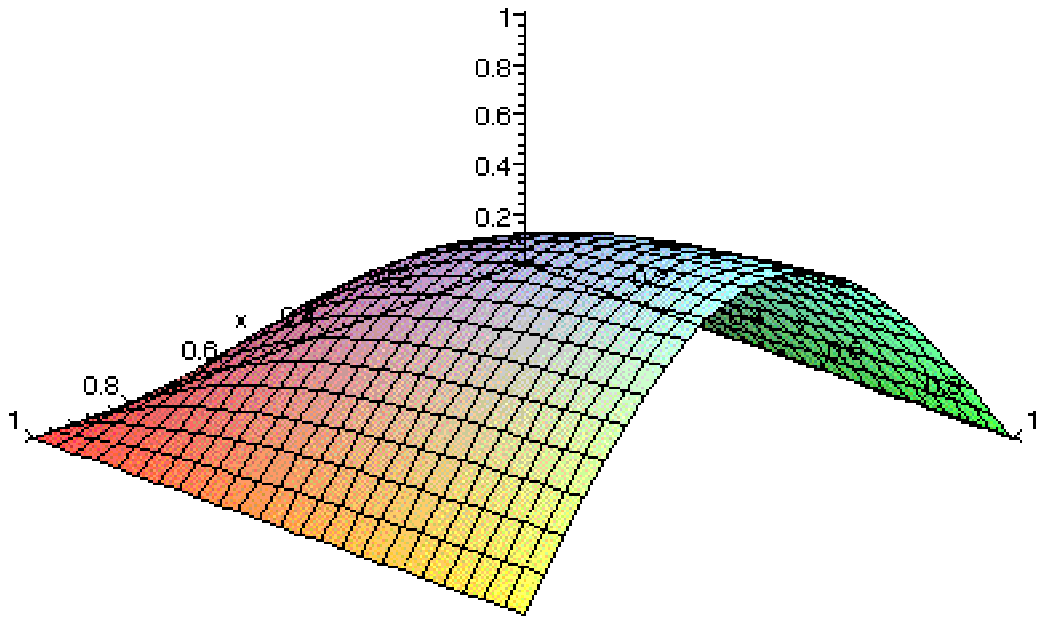
Picture of steady state:

$$\sinh(\pi(1 - y))\,\sin(\pi x)/\sinh(\pi)$$



The function w will be the transient solution

$$w(0, x, y) = \sin(\pi x) - \sinh(\pi(1-y))\sin(\pi x)/\sinh(\pi)$$



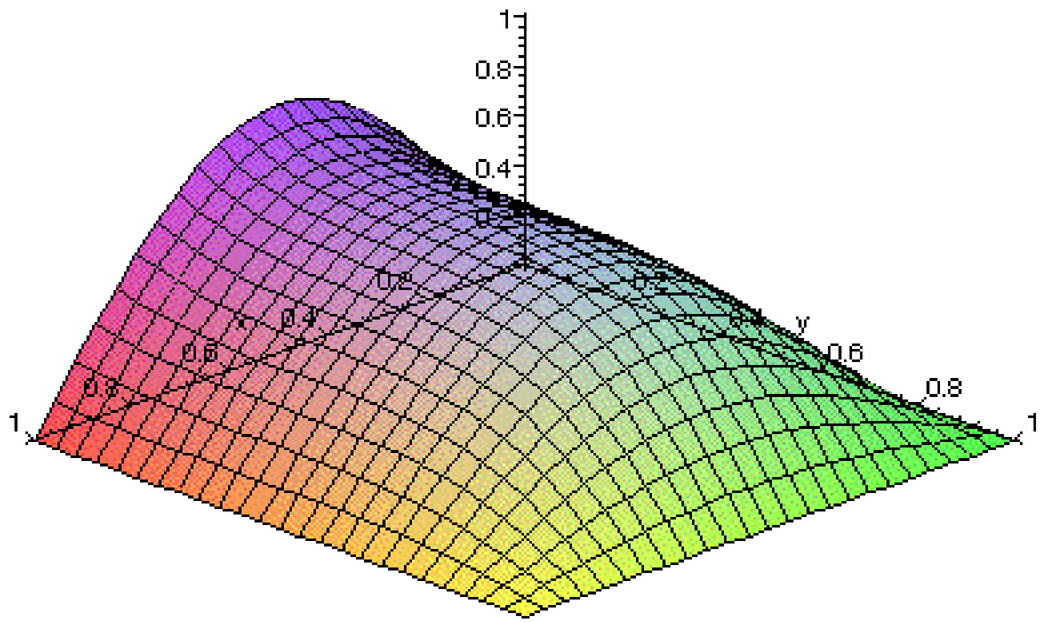
Make the transient solution w.

$W(t, x, y) =$

$$\Sigma_n \Sigma_m A_{nm} \sin(n\pi x/a) \sin(m\pi y/b) \exp(-((n/a)^2+(m/b)^2)\pi^2 t)$$

$$u(t, x, y) = w(t, x, y) + v(x, y)$$

Graph of u(1/50, x, y)



Example: Speed of Diffusion.

If we want to incorporate speed of diffusion, we should change the model by including c :

$$u_t(t, x, y) = c\,(u_{xx} + u_{yy})$$

This time, we choose conditions

$$u(t, x, 0) = 100, \quad u(t, x, 1) = 100$$

$$u(t, 0, y) = 100, \quad u(t, 1, y) = 100$$

and initial conditions

$$u(0, x, y) = 0.$$

We ask: for various values of c, what is the value of t such that $u(t, 1/2, 1/2) = 50$?

Is it clear that the steady state solutions is 100? And, that general solution is

$$u(t, x, y) =$$

$$\Sigma_n \Sigma_m A_{nm} \sin(n\pi x/a) \sin(m\pi y/b) \exp(-c ((n/a)^2+(m/b)^2)\pi^2 t)$$

$$+ 100$$

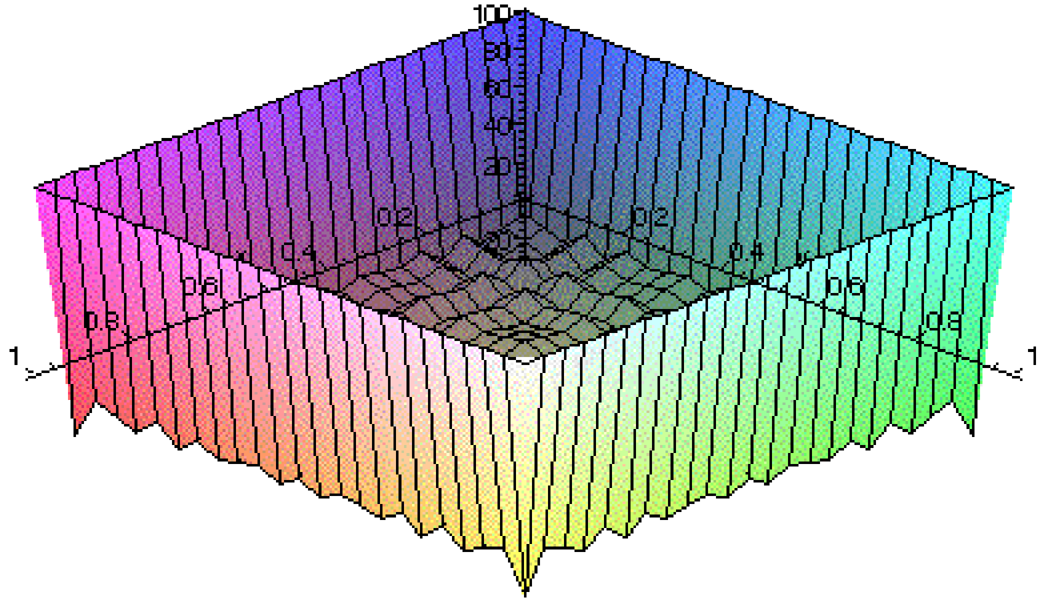
where A_{nm} is a double integral.

$$A_{nm} = 4\iint w(0,x,y)\sin(nx\pi)\sin(my\pi)\,dxdy/ab$$

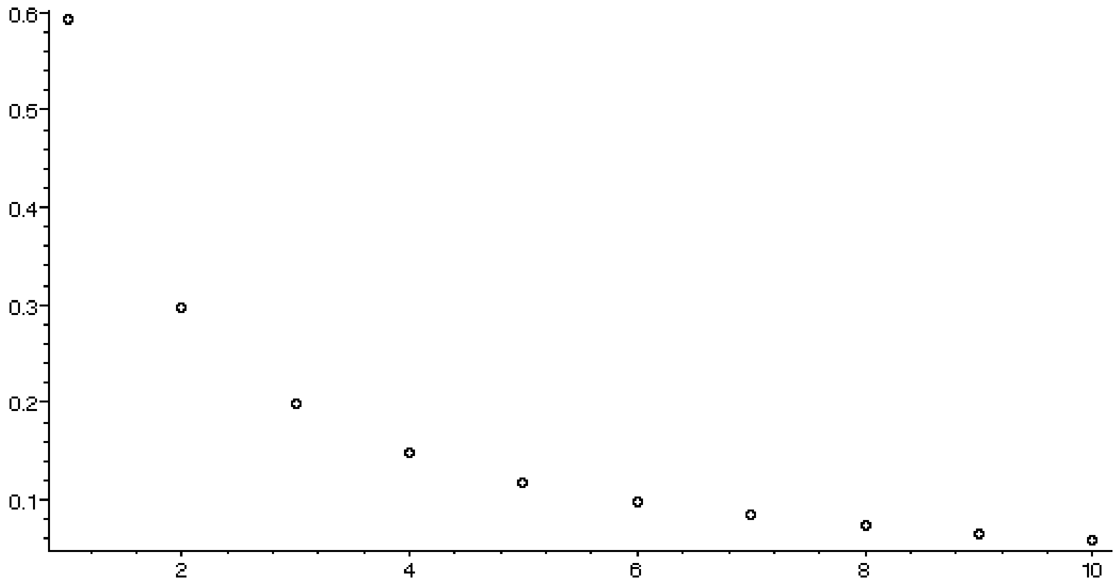
$$= -4\iint 100\sin(nx\pi)\sin(my\pi)\,dxdy/ab$$

$$= -400\,(\cos(n\pi)-1)\,(\cos(m\pi)-1)/(n\pi m\pi);.$$

Graph of u(0, x, y):



Compute c's. Solve: u(c, t, 1/2, 1/2) = 50.
Graph is for c = p/10.



Assignment: See Maple Worksheet.

In this Module 32, we have worked a diffusion problem on a rectangle. Also, we have observed the effect which variations in c have on the speed of diffusion.