Module 33: Heat Equation on a Disk

We consider the diffusion equation on a disk.

The problem can be written as

$$1/k \; u_t = \Delta u,$$

with $u(t, a, \theta) = f(\theta)$, $0 < t$, $-\pi < \theta < \pi$,

and $u(0, r, \theta) = g(r, \theta)$, $0 \le r < a$, $-\pi < \theta < \pi$.

To do this problem, we first solve the steady state problem

$$0 = \Delta v, \quad v(a, \theta) = f(\theta), \; -\pi < \theta < \pi.$$

Then solve the transient problem

$$1/k \; u_t = \Delta u,$$

with $u(t, a, \theta) = 0, \quad 0 < t, \; -\pi < \theta < \pi$ and
$u(0, r, \theta) = g(r, \theta) - v(r, \theta) \quad 0 \leq r < a, \; -\pi < \theta < \pi$.

Finally, the solution for the original problem is the sum of these two solutions.

First problem: See Modules 30 and 31.

Second Problem:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \Theta^2} = \frac{\partial u}{\partial t}$$

Separation of variables leads to

(1) $\Theta'' + \mu^2 \Theta = 0$, $\Theta(\pi) = \Theta(-\pi)$, $\Theta'(\pi) = \Theta'(-\pi)$.

(2) $T' = -\lambda^2 T$, and

(3) $r(rR')' - \mu^2 R = -\lambda^2 r^2 R$, $R(a) = 0$, R is bounded on $[0, a)$.

Equation 3 leads to a study of Bessel's functions.

An example, independent of Θ:

$$u_t = \Delta u,$$

with $u(t, a, \Theta) = 0, \quad 0 < t, \quad -\pi < \Theta < \pi,$

and $u(0, r, \Theta) = g(r), \quad 0 \le r < a, \quad -\pi < \Theta < \pi.$

Is it clear that the solution of this equation will be independent of Θ? Thus,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) = \frac{\partial u}{\partial t}$$

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Result: two differential equations.

(1) $T' + \lambda^2 T = 0$, and

(2) $r(rR')' + \lambda^2 r^2 R = 0$, with $R(a) = 0$, and R bounded on $[0, a)$.

We examine the second differential equation. Expanded, it is

$$r^2 R'' + rR' + \lambda^2 r^2 R = 0.$$

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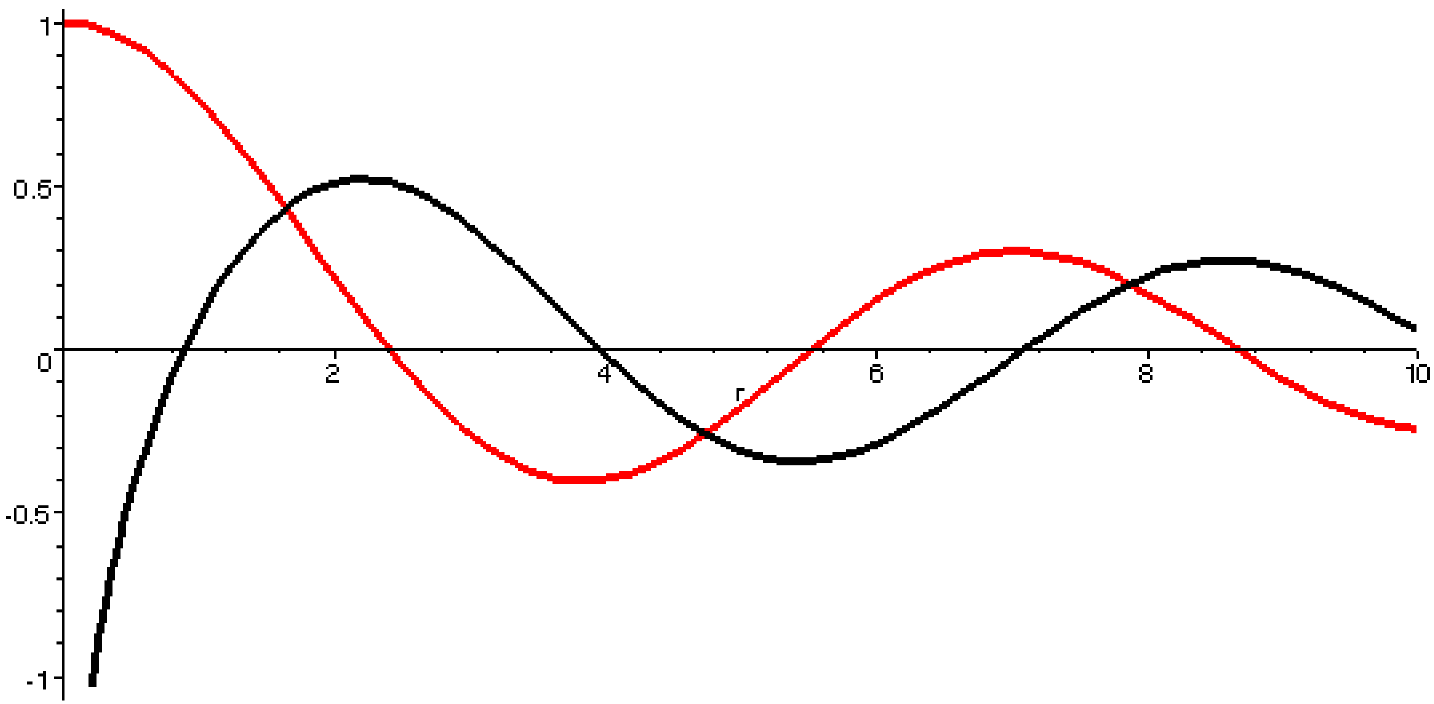
The solutions are Bessel functions.



There are several observations to make:

(1) BesselJ(0, r) is one at zero and BesselY(0, r) is unbounded at zero.

(2) The two solutions oscillate, are not periodic, and have intertwining zeros.

(3) These form eigenfunctions corresponding to the eigenvalues $-\lambda^2$ for the differential operator

$$L(R) = 1/r \; (r R ') '.$$

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This last means that if R(r) = Bessel(λ r) then

$$L(R) = -\lambda^2 R.$$

Recall that the differential equation of interest was

$$r (r R ') ' + \lambda^2 r^2 R = 0$$

and the boundary condition was R(a) = 0.

It is of value to think of the comparison of these two functions with the cosine and sine functions. We are interested in bounded solutions on the disk, set aside the unbounded Bessel function.

Suppose $a > 0$. Take $J(r)$ to be the bounded Bessel function. (Compare with sine.)

Comparison 1: There is an infinite increasing sequence $\lambda_1, \lambda_2, \lambda_3, \ldots$ such that $J(\lambda_p a) = 0$. For the sine function, these numbers were $n\pi/a$.

Comparison 2: The numbers $-\lambda_j^2$ are eigenvalues and $R(r) = J(\lambda_j r)$ is an eigenfunction for the operator $L(R)$ defined by

$$L(R) = 1/r \; (r R ') '$$

with $R(a) = 0$.

The function $\sin(n \pi/a \; x)$ was an eigenfunction for $L(\Theta) = \Theta''$ with $\Theta(a) = 0$.

Comparison 3: Orthogonality persists in the sense that

$$\int_0^a J(\lambda_j x)\, J(\lambda_k x)\, x\, dx = 0 \text{ if } i \neq j,$$

and

$$\int_0^a J(\lambda_j x)\, J(\lambda_j x)\, x\, dx = a^2/2 \;\; J'(\lambda_j a)^2 .$$

Recall the situation with $\sin(n\pi/a\, x)$.

Comparison 4: If f(r) is sectionally smooth on the interval (0, a) then

$$f(r) = \sum_{n} a_n J(\lambda_n r)$$

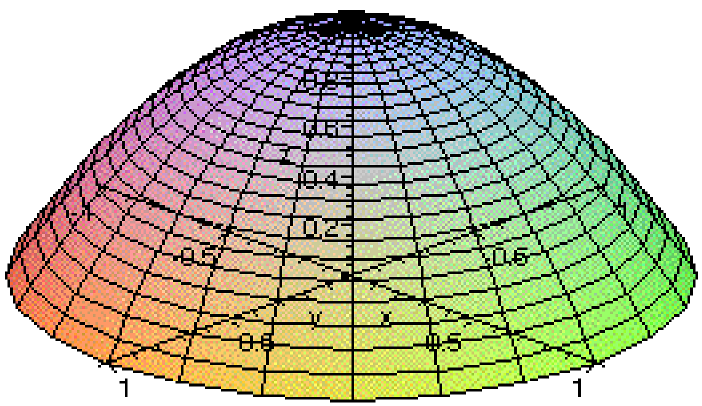
where $a_n = \langle f(r), J(\lambda_n r) \rangle / \langle J(\lambda_n r), J(\lambda_n r) \rangle$

In this case, the dotproduct is defined by

$$\langle f, g \rangle = \int_0^a f(r)\, g(r)\, r\, dr .$$

It seems appropriate to do an example:

$$u(0, r, \theta) = 1 - r^2 \,, \quad \text{independent of } \theta.$$



We need the zeros of the Bessel function J. There are several ways to get them. Successive approximation, or Maple knows them.

zero[1] := 2.404825558

zero[2] := 5.520078110

zero[3] := 8.653727913

zero[4] := 11.79153444

zero[5] := 14.93091771

zero[6] := 18.07106397

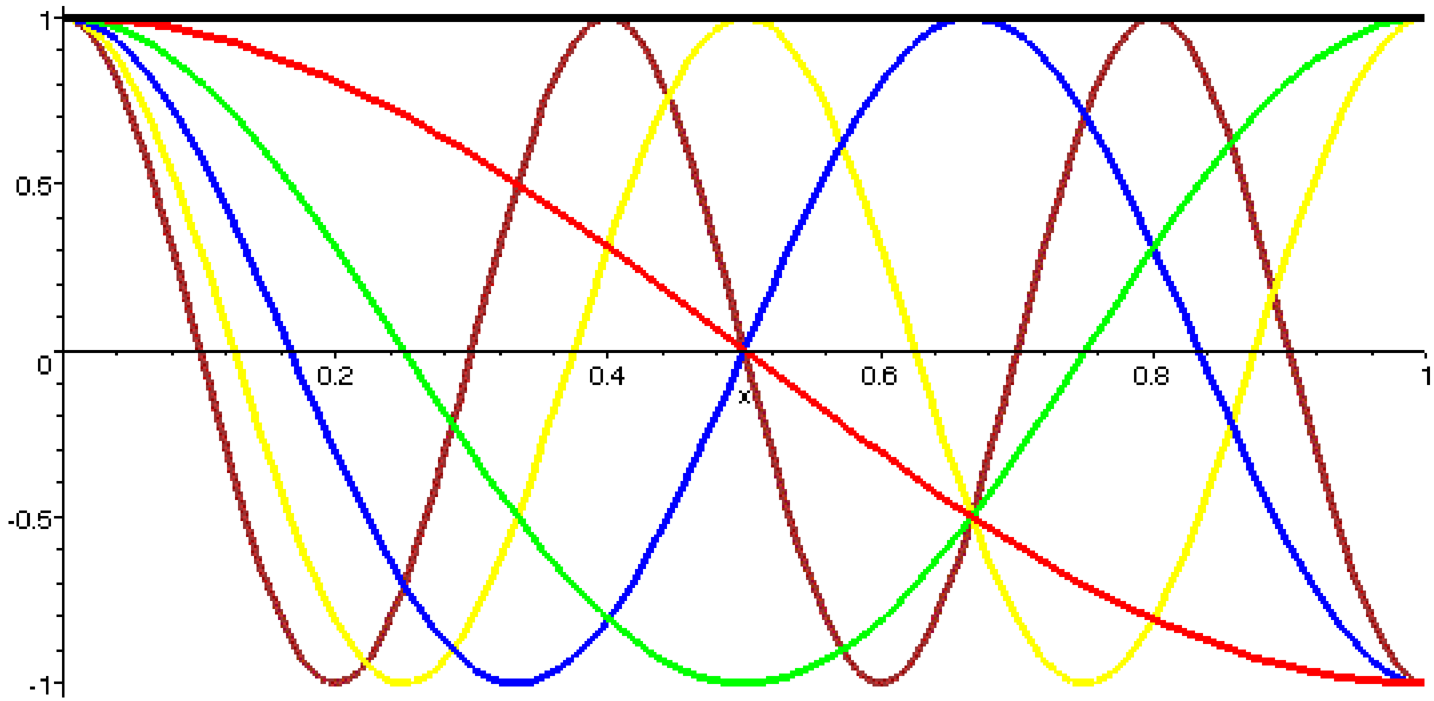
zero[7] := 21.21163663

zero[8] := 24.35247153

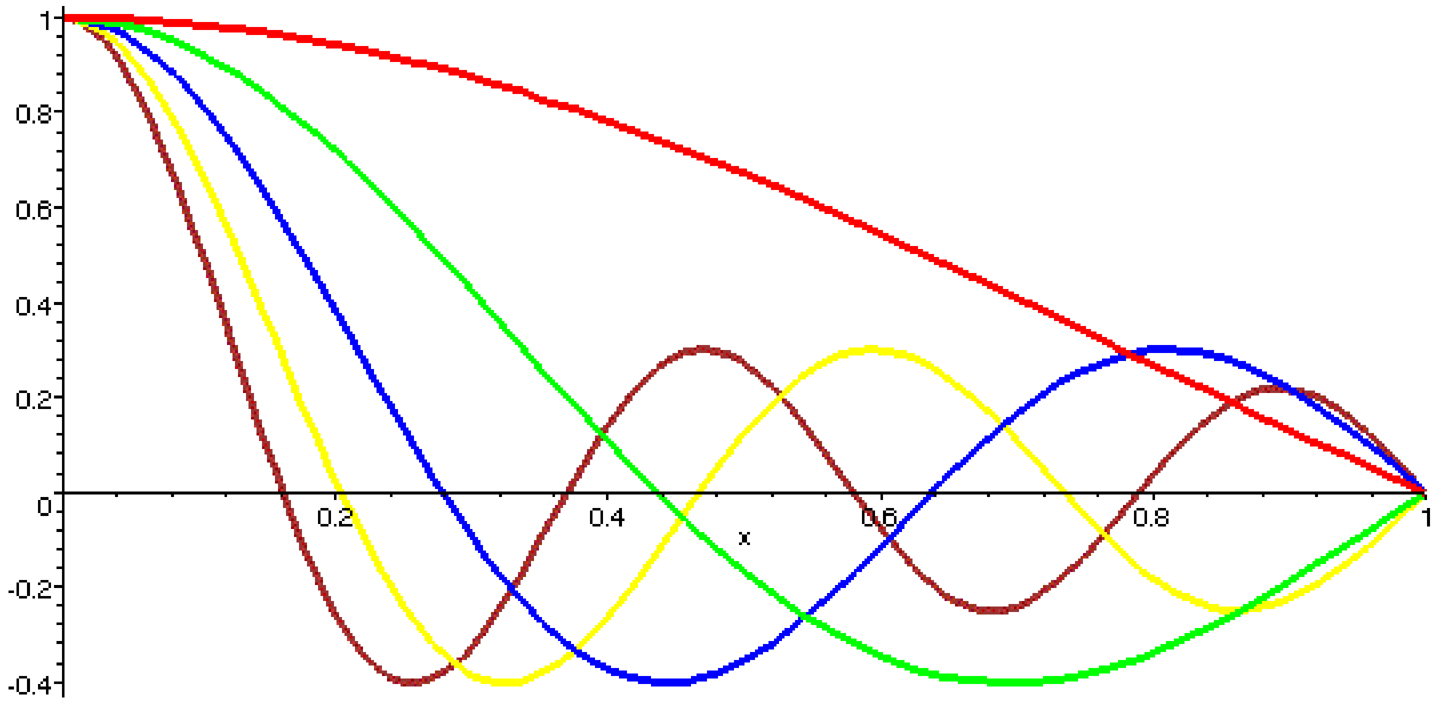
zero[9] := 27.49347913

zero[10] := 30.63460647

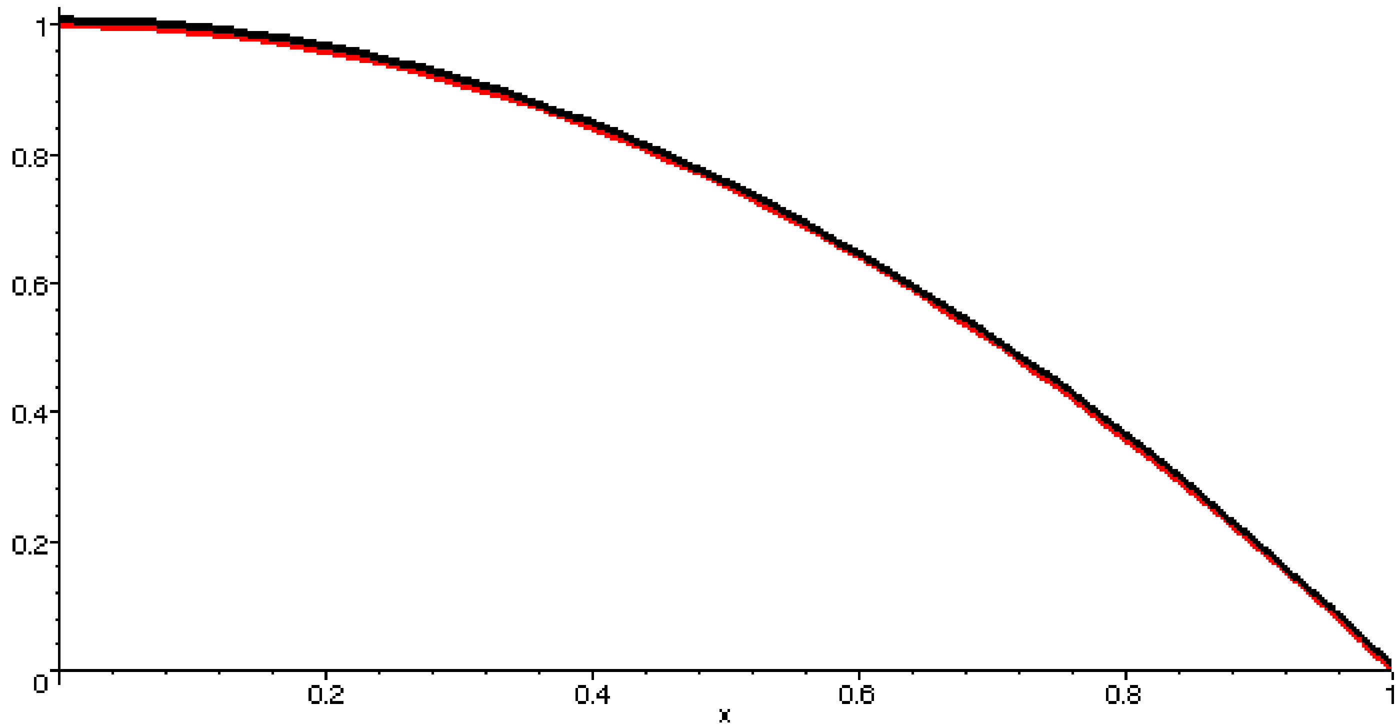
The first six cos(n π x) graphs.

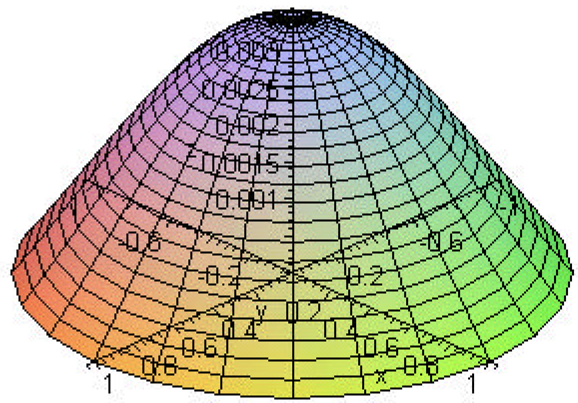


The first five J(zero[n] x).



Bessel series approximation for $1 - r^2$.





$u(1,r)$ where $u(t, r) = \Sigma_n a_n J(\lambda_n r) \exp(-\lambda_n^2 t)$.

Assignment. See the Maple worksheet. I think you will like the problem.

In this Module 33, we have
(1) introduced Bessel functions and
(2) solved a diffusion equation on a disk.