Module 34: Recipe for a Cheese Cake

Here is a recipe for a cheese cake. If you have to worry about cholesterol, just stay away from this one.

If you don't, this one is really great. Fix it a day before you want to impress guests and serve it with the best coffee you have. Wow!

Guaranteed a success if you follow the directions.

Ingredients: Herod's Cheese Cake

2 1/2 lb. cream cheese (five 8 oz. pkg.)

1 3/4 cup sugar

3 t. flour

1 1/2 t. grated orange rind

1 1/2 t. grated lemon rind

1/4 t. vanilla

5 eggs

2 egg yolks

1/4 c. heavy cream.

Put cheese in mixer bowl. Beat at low speed. Add sugar gradually, then the remainder of ingredients in order. (Eggs should be added one at a time.) When blended and smooth, pour in a lined pan and place in a pre-heated 550 degree oven. Bake 12 - 15 minutes. Reduce heat to 200 degrees. (Cool oven quickly by leaving the door open and fanning as necessary.) Continue baking for 1 hour. Cool before cutting. The cake is better after refrigeration. Lasts indefinitely in the refrigerator ... if you can resist it.

The Mathematical Question

As an applied mathematician, here is the question: Eggs congeal at about 140 degrees. All the ingredients of the cheese cake before cooking are about 46 degrees. When the ingredients are mixed and placed in the hot oven, how long does it take to get the center of the cooking cake to 140 degrees?

The Mathematical Model.

This is a heat diffusion problem. We model it as

$$du(t, r, \Theta, z)/dt = c\,\Delta u$$

Side boundary: $u(t, 5, \Theta, z) = 550$,

$t > 0$, $-\pi < \Theta < \pi$, $0 < z < 1.5$,

Top and bottom boundary:

$u(t, r, \Theta, 0) = 550 = u(t, r, \Theta, 1.5)$,

Initial condition: $u(0, r, \Theta, z) = 46$.

We ask: at what time is the temperature in the middle 140 degrees? That is, compute t so that

$$u(t, 0, 0, 3/4) = 140.$$

Solution:

Recall that when the Laplacian operator is written out for a cylinder the equation is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \Theta^2} + \frac{\partial^2 u}{\partial z^2} = 1/c\ \frac{\partial u}{\partial t}$$

The number c is the rate of diffusion for heat through the cream cheese/egg mixture. For this problem, since the initial condition and boundary condition are independent of theta, the solution will be independent of theta. Thus, we can rewrite the partial differential equation as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)+\frac{\partial^2 u}{\partial z^2} = 1/c \; \frac{\partial u}{\partial t}$$

Agree that the steady state solution for the equation will be 550 degrees. (As a cook, you must not let the cake achieve this state!)

Thus, we solve the problem with homogeneous boundary conditions and add 550. For ease of typing, take a = 5, the radius of the cake, and b = 1.5, the height of the cake.

Perform a separation of variables on u. That is, assume that u(t, r, z) = T(t) R(r) Z(z). We get that

$$1/r\ (r\ R\ ')\ '\ Z\ T + Z\ ''\ R\ T = 1/c\ \ T\ '\ R\ Z.$$

Dividing by R Z T gives

$$1/r\ \ (r\ R\ ')\ '\ /\ R\ \ +\ \ Z\ ''\ /\ Z\ \ =\ \ 1/c\ \ T\ '\ /\ T.$$

As usual, we make the argument that each of the terms of the sum on the left side of the above equation must be constant. We have that

(1) $1/r\, (r\, R\,')\,' \;=\; -\mu^2 R\,,\;\; R(a) = 0.$

(2) $Z\,'' = -\lambda^2 Z,\;\; Z(0) = Z(b) = 0.$

and

(3) $T\,' = -\, c\, (\mu^2 + \lambda^2)\, T.$

We handle these one at a time.

Analysis of equation (1):

We rewrite equation (1) as

$$(r\,R')' = -\mu^2\, r\, R\,, \quad R(a) = 0.$$

or

$$r^2 R'' + r R' + \mu^2 r^2 R = 0, \text{ with } R(a) = 0.$$

The solution for this last equation is among the Bessel functions.

$$R(r) = \mathrm{BesselJ}(0, \mu\, r).$$

Analysis of equation (2):

Equation (2) is recognized as an equation, with boundary conditions, which defines the sine functions.

$$Z(z) = \sin(\lambda z).$$

Analysis of equation (3):

Equation (3) is recognized as an equation for the exponential function.

$$T(t) = \exp(-c\,(\mu^2 + \lambda^2)\,t).$$

Products of solutions of (1), (2), and (3).

Products of solutions for equations (1), (2), and (3) should form a solution for the original equation.

$$u(t, r, z) = \text{BesselJ}(0, \mu r) \sin(\lambda z) \exp(-c(\mu^2 + \lambda^2) t)$$

(Recall why we use BesselJ and not BesselY.)

General solution

We add constants times products of solutions for (1), (2), and (3) to make the general solution of the original equation:

$$u(t, r, z) =$$

$$\sum_{m} \sum_{n} A_{n,m} \text{BesselJ}(0, \mu r) \sin(\lambda z) \exp(-c(\mu^2 + \lambda^2) t)$$

For λ_n , we have that $\sin(\lambda_n\ b) = 0$, so that

$$\lambda_n\ b = n\ \pi.$$

```
> for n from 1 to 40 do
>       λ[n]:=n π/(3/2):
> od:
```

For μ_m , we have that BesselJ(0, μ_m a) = 0, so that μ_m is the m[th] zero of BesselJ(0, x) divided by a.

```
> for m from 1 to 40 do
>     μ[m] :=evalf(BesselJZeros(0, m))/5:
> od:
```

Computing coefficients

We choose the coefficients $A_{m,n}$ so that when $t = 0$, $46 - 550 =$

$$\sum_m \sum_n A_{n,m}\, \mathrm{BesselJ}(0, \mu_m r)\, \sin(\lambda_n z)$$

These coefficients will come from the Fourier coefficients:

$$A_{n,m} = (46\text{-}550) \int_0^b \sin(\lambda_n z)\, dz \int_0^a R(\mu_m r)\, r\, dr \Big/ \int_0^b \sin(\lambda_n z)^2\, dz \int_0^a R(\mu_m r)^2\, r\, dr$$

where $R(r) = \text{BesselJ}(0, r)$

To speed the calculations, we do all the computations for n and then all the computations for m and multiply these.

Think about why this is better programming.

The sine terms:

```
> for n from 1 to 40 do
> T[n]:=int(sin(lambda[n]*z),z = 0 .. 3/2)/
>     int(sin(lambda[n]*z)^2,z = 0 .. 3/2):
> od:
```

The Bessel terms.

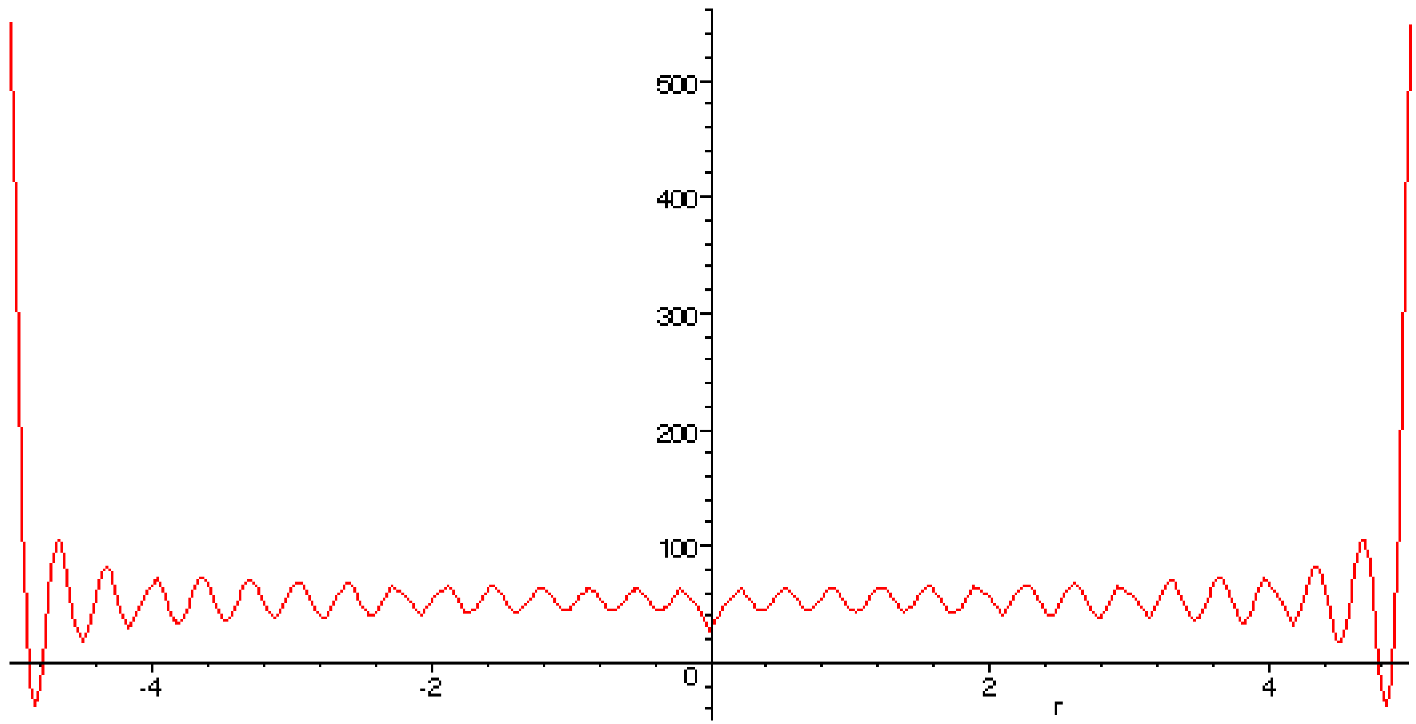
```
> for m from 1 to 30 do
> B[m]:=int(BesselJ(0,mu[m]*r)*r,r = 0 .. 5)/
>        int(BesselJ(0,mu[m]*r)^2*r,r = 0 .. 5):
> od:
```

Here is multiplying these together to get $A_{n,m}$.

```
> for n from 1 to 40 do
> for m from 1 to 30 do
>    A[n,m]:=(46-550)*T[n]*B[m]:
> od: od:
```

From here on, the program takes a lot of space. Some how, this is not a surprise: if you eat much of this cheese cake, you will take a lot of space!

A graph for an approximation of the initial value



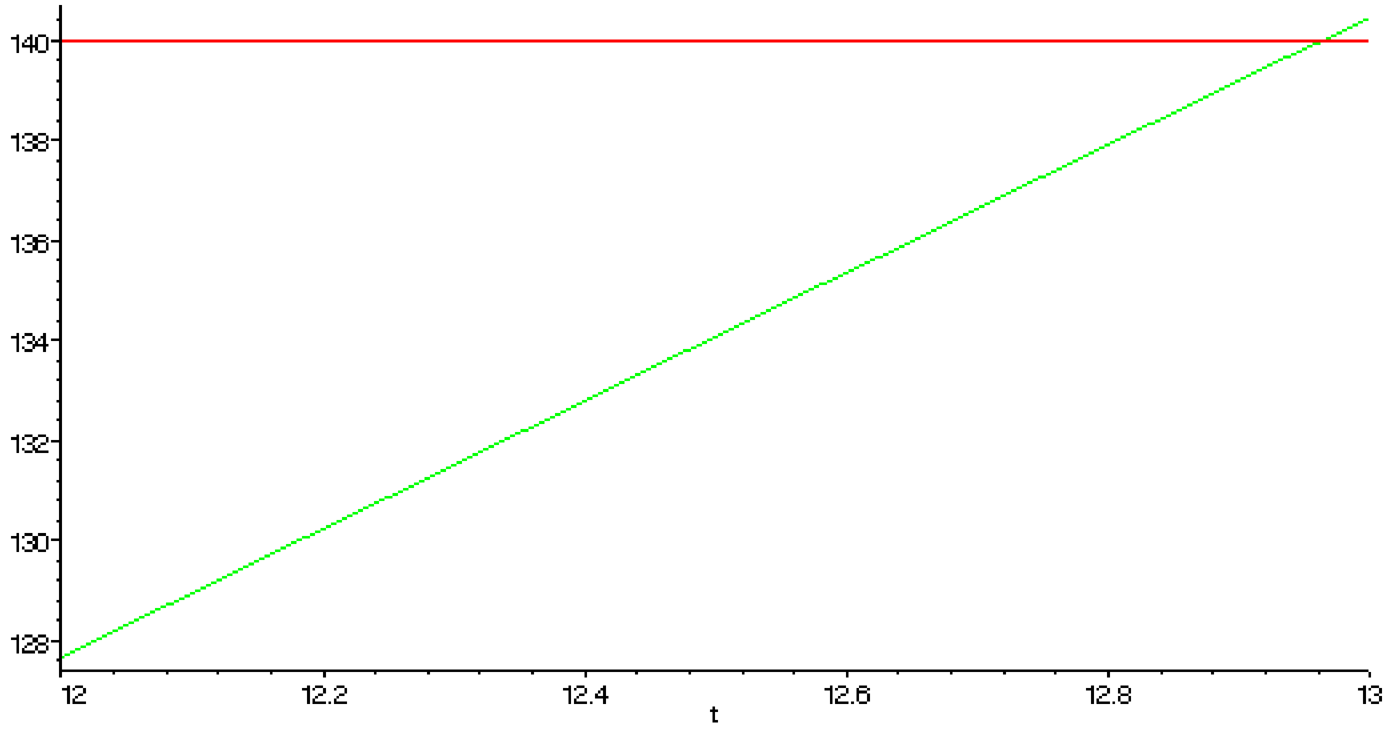
$$u(t, r, z) =$$

$$\sum_{m} \sum_{n} A_{n,m} \operatorname{BesselJ}(0, \mu r) \sin(\lambda z) \exp(-c (\mu^2 + \lambda^2) t)$$

We are ready to find when the middle of the cake achieves 140. To do this, you need to know c for this cheese cake. It is

```
> c:=0.0077;
```

A graphical method for finding when the center is 140 degrees.



A numerical procedure for finding t.

```
> fsolve(u(t,0,3/4)+550=140,t,12.9..13);
```

Assignment: See the Maple worksheet.

In this Module 34, we have worked a diffusion problem in a cylinder with initial conditions independent of Θ.