Module 35: Warm Spheres

An important coordinate system is that of a sphere. In that coordinate system we have

ρ is the distance from the origin,

Θ is the angle in the x - y plane; that is, it measures longitude.

Φ is the angle from the top; that is, it measures latitude.

Why is $\rho = 1$, $0 < \Theta < 2\pi$, $0 < \Phi < \pi$ a sphere with radius 1 ? Recall the connection.

The connection with rectangular coordinates is

$x = \rho \sin(\Phi) \cos(\Theta)$

$y = \rho \sin(\Phi) \sin(\Theta)$

$z = \rho \cos(\Phi)$

That the surface $\rho = 1$ is a sphere follows from

$$x^2 + y^2 + z^2 = 1.$$

Also, $\rho = 1$, $\Theta = \pi / 4$; , $0 < \Phi < \pi$ is a half circle running from the north pole to the south pole of the sphere.

Finally, $\rho = 1$, $\Phi = 49$ degrees, defines a part of the boundary between Western Canada and Western United States.

In this coordinate system, the Laplacian Operator is

$$1/\rho^2\{ \partial/\partial\rho(\rho^2\, \partial u/\partial\rho)+1/\sin(\varphi)\, \partial/\partial\varphi(\sin(\varphi)\, \partial u/\partial\varphi)$$
$$+ 1/\sin(\varphi)^2\, \partial^2 u/\partial\theta^2 \}$$

We solve $0 = \Delta u$, $u(1, \Phi) = f(\Phi)$, where u is independent of Θ .

From the assumption that

$$u(\rho, \varphi) = R(\rho)\,\Phi(\varphi),$$

we are led to this separation of variables situation:

$$(\rho^2 R'(\rho))' / R + (\sin(\varphi)\Phi')'/\sin(\varphi)\,\Phi = 0.$$

This suggests the ordinary differential equations

$$(\rho^2 R'(\rho))' - \mu^2 R(\rho) = 0, \quad 0 < \rho < 1,$$

$$(\sin(\varphi)\Phi')' + \mu^2 \sin(\varphi)\,\Phi = 0, \quad 0 < \varphi < \pi.$$

$$(\rho^2 R'(\rho))' - \mu^2 R(\rho) = 0, \quad 0 < \rho < 1,$$

$$(\sin(\varphi)\Phi')' + \mu^2 \sin(\varphi)\,\Phi = 0, \quad 0 < \varphi < \pi .$$

Neither equation has a boundary condition. We have conditions of boundness. In the second equation, we take $x = \cos(\varphi)$, $y(x) = \Phi(\varphi)$.

Hence,

$$(\sin(\varphi)\Phi')' = \sin(\varphi)^3 y''(x) - 2\sin(\varphi)\cos(\varphi)\,y'(x).$$

The second differential equation above becomes

$$\sin(\varphi)^2 y''(x) - 2\cos(\varphi)\,y'(x) + \mu^2 y(x) = 0.$$

In terms of x alone,

$$\sin(\varphi)^2 y''(x) - 2\cos(\varphi)\, y'(x) + \mu^2 y(x) = 0$$

becomes

$$(1-x^2)\, y''(x) - 2\, x\, y'(x) + \mu^2 y(x) = 0, \; -1 < x < 1.$$

We are led to consider Legendre Polynomials.
We digress to recall these functions.

A recollection of Legendre Polynomials

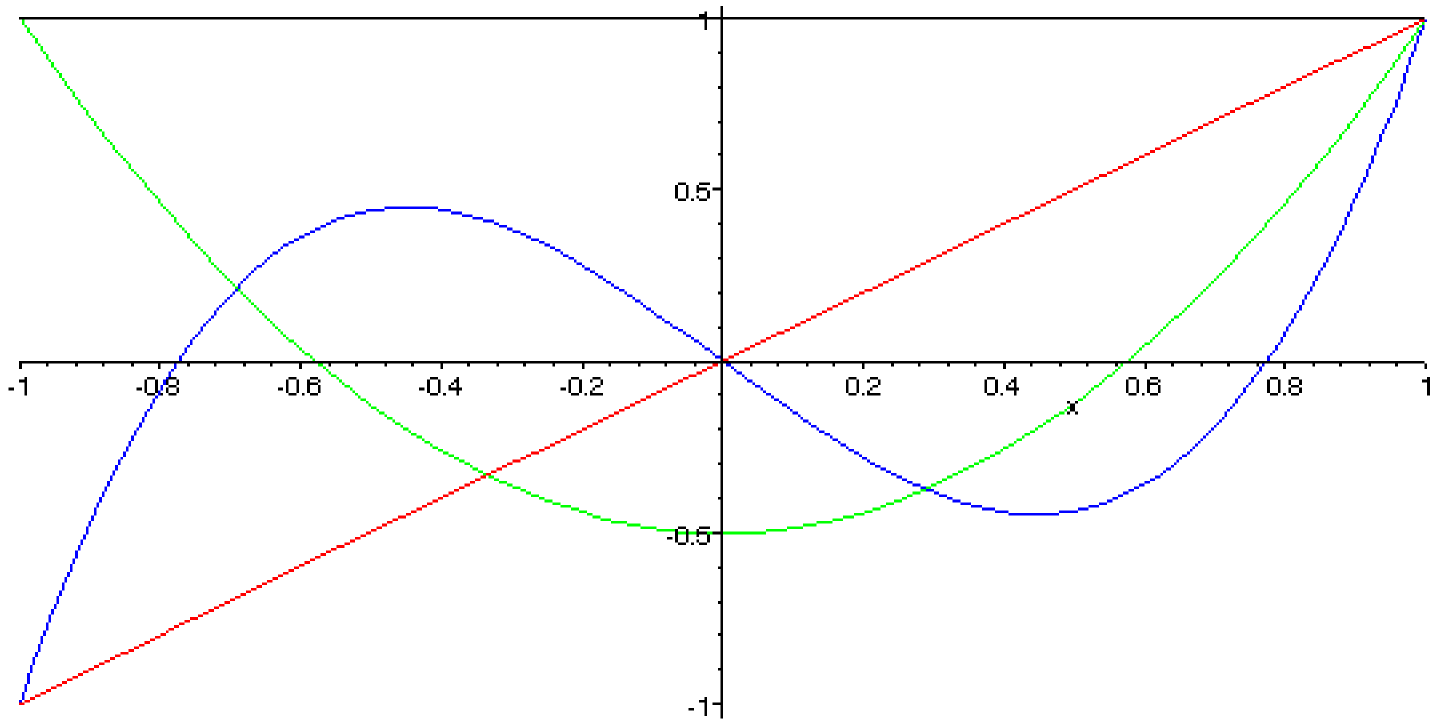
Here are three ways to conceive of the Legendre Polynomials -- four, if we include Maple.

Method 1: solve the differential equation

$$(1-x^2)\, y''(x) - 2\, x\, y\,'(x) + \mu^2\, y(x) = 0, \quad -1 < x < 1.$$

Take $\mu^2 = n\,(n+1)$.

Here are graphs of 4 Legendre Polynomials



Method 2: We could apply the Gramm Schmidt Process to 1, x, x^2, ...
We examined these ideas earlier.

Recall that

$$\int_{-1}^{1} Pn(x)\, Pm(x)\, dx = 0 \text{ if n is not m.}$$

Method 3. We could generate these by taking the appropriate derivatives.

We show that

$$\frac{1}{2^n\, n!} \; d^n (x^2 - 1)^n / dx^n$$

is the n^{th} polynomial.

Method 4: Use the recursion formulas

$(n+1)\ P_{n+1}(x) + n\ P_{n-1}(x) = (2\ n + 1)\ P_n(x)\ x.$

With this method, we assume you know the first two.

$P_0(x) = 1, \ P_1(x) = x,$

$2\ P_2(x) + 1\ P_0(x) = 3\ P_1(x)\ x,$ or

$2\ P_2\ (x) + 1 = 3\ x^2.$

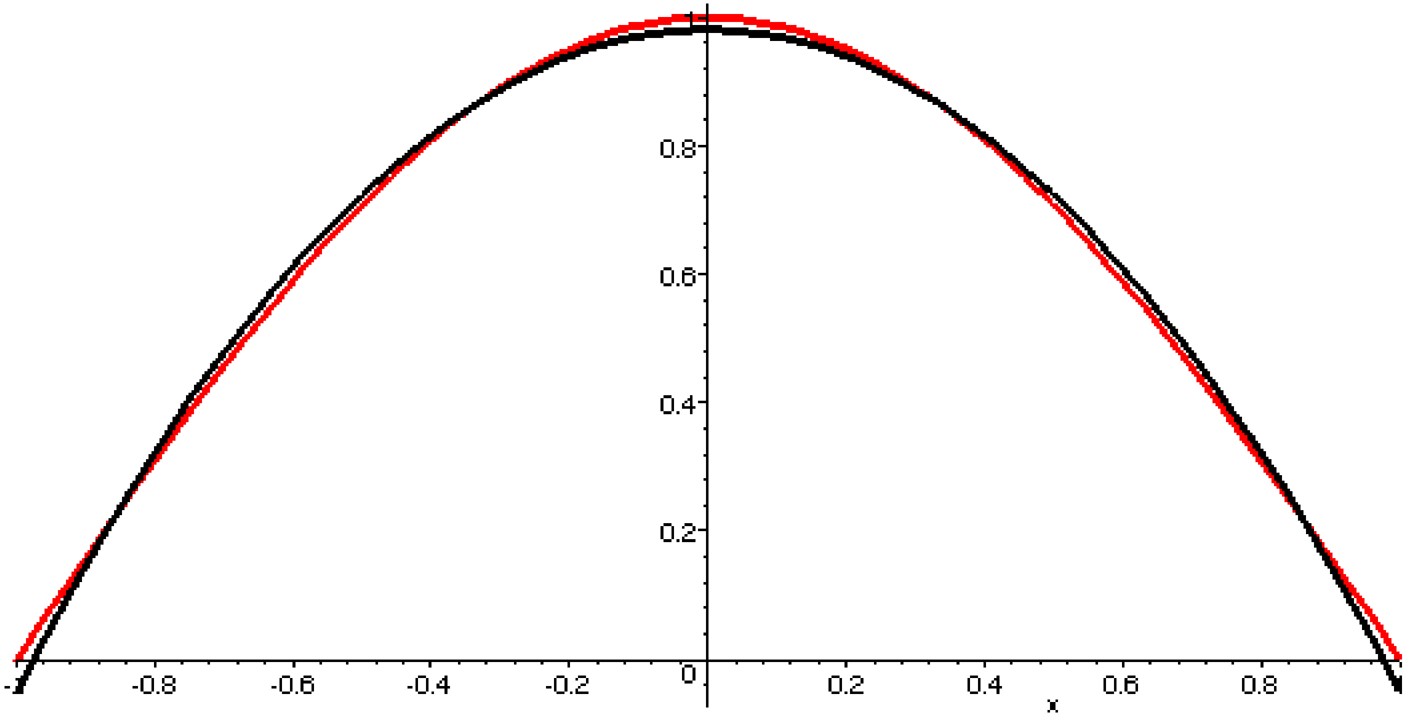
Observation 1: if $0 < m < n$, then

$$\int_{-1}^{1} x^m P_n(x)\, dx = 0.$$

Observation 2: We have a formula for the norm of $P_n(x)$:

$$\int_{-1}^{1} P_n(x)^2\, dx = 2/(2n + 1).$$

Observation 3: We can make polynomial approximations for functions on [-1, 1].



Recall where we were. The partial differential equation led to two ordinary differential equations.

The second differential equation

$$\sin(\varphi)^2 y''(x) - 2 \cos(\varphi)\, y'(x) + \mu^2 y(x) = 0$$

became

$$(1-x^2)\, y''(x) - 2\, x\, y'(x) + \mu^2 y(x) = 0, \ -1 < x < 1.$$

This led to

$(1-x^2)\, y''(x) - 2\, x\, y\, '(x) + \; n\, (n+1)\, y(x) = 0,$

and Legendre Polynomials.

Here is the first of the differential equations from above, with this $\mu^2 = n\,(n+1)$:

$(\,\rho^2\, R\, '(\rho))' - n\,(n+1)R(\rho) = 0, \;\; 0 < \rho < 1,$

$\rho^2\, R'' + 2\,\rho\, R\, ' - n\,(n+1)\, R(\rho) = 0, \;\; 0 < \rho < 1.$

$\rho^2 R'' + 2 \rho R' - n(n+1) R(\rho) = 0, \quad 0 < \rho < 1.$

has bounded solution r^n.

First, we verify that sums of products of solutions are solutions.

$$u(r, \varphi) = \sum_n a_n r^n P_n(\cos(\varphi))$$

is a general solution for

$$u(r, \varphi) = \Delta u$$

We are now ready to compute the coefficients for a boundary conditon. We solve

$$0 = \Delta u \text{ with } 0 < \rho < 1, \; 0 < \varphi < \pi, \; 0 < \theta < 2\pi$$

with boundary condition $u(1, \varphi) = f(\varphi)$.
The coefficients will be

$$a_n = (2n+1)/2 \int_{-1}^{1} f(\varphi)\, P_n(\cos(\varphi))\, d\varphi.$$

Let's take the special case that $f(\varphi) = \cos(\varphi)$.
In this case only a_1 is not zero.

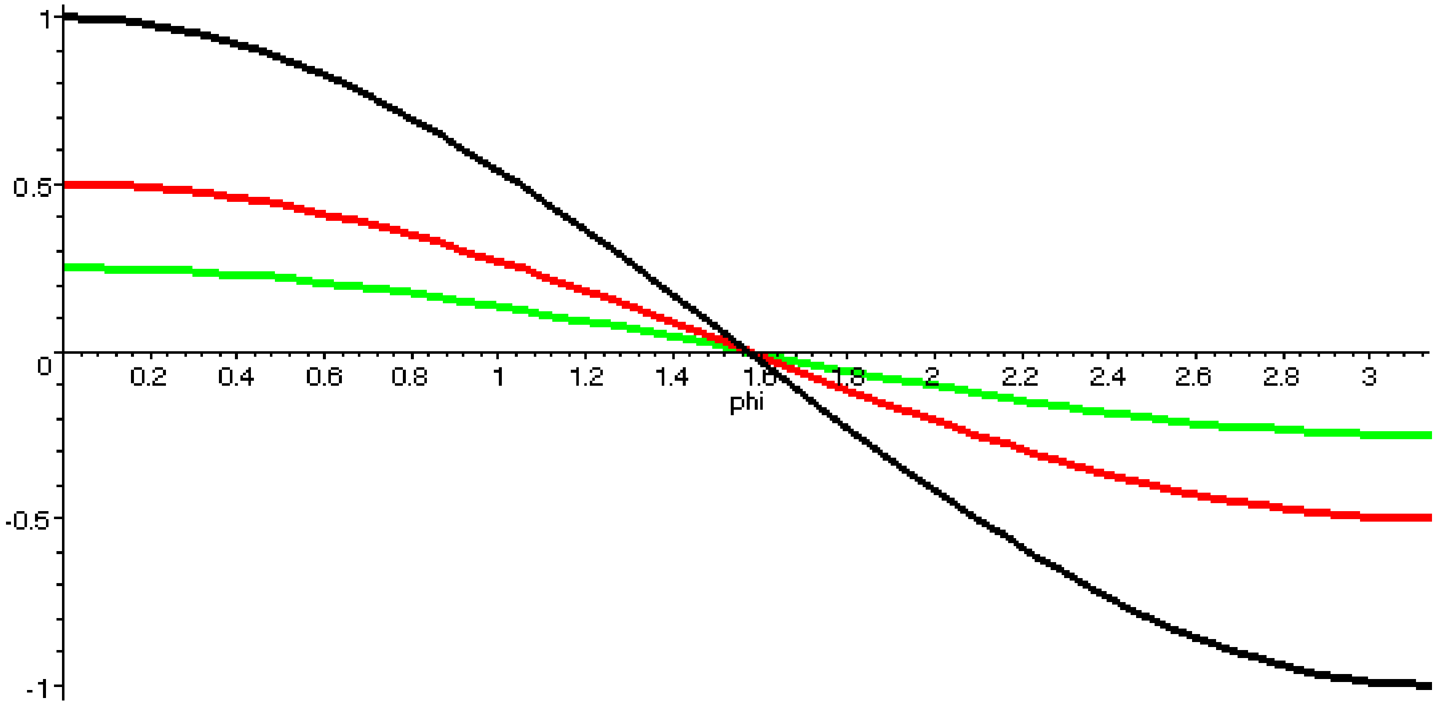
Thus, the solution for this problem is

$$u(r, \varphi) = r \cos(\varphi).$$

How can we illustrate what we have?

(1) Each cross sectional plane parallel to the x-y plane has value z.

How can we illustrate what we have?



(2) Hold r fixed and let φ go from 0 to π. See the graph.

Assignment: See the Maple worksheet.

In this Module 35, we have solve the Laplaces equation on a sphere. To do this, we had to rewrite the Laplacian operator in spherical coordinates. Use was made of Legendre Polynomials.