Module 36: Vibrations of a Circular Drum

A standard problem: a vibrating circular membrane. Here, we treat the simple case in which the initial conditions are independent of θ. We recall the wave equation:

$$1/r\ \partial(r\ \partial u/\partial r)/\partial r + 1/r^2\ \partial^2 u/\partial\theta^2 = 1/c^2\ \partial^2 u/\partial t^2.$$

Except, in this simplified case with the initial conditions independent of θ, we can drop the second term of the left side.

Thus, we have

$$\frac{1}{r}\,\partial(r\,\partial u/\partial r)/\partial r = 1/c^2\,\partial^2 u/\partial t^2.$$

Boundary conditions will be

$u(t, a) = 0$, because the sides of the drum are nailed down,

$u(0, r) = f(r)$, which is the initial displacement, and

$\partial u/\partial t(0, r) = g(r)$, which is the initial velocity.

At this point in our study of partial differential equations, we know to start by separation of variables. Assume that

$$u(t, r) = T(t)\,R(r).$$

Using this assumption in the partial differential equation leads to

$$1/r\ \ (r\,R\,')\,'\ T = 1/c^2\ \ R\ T\,''$$

which, in turn, leads to the two ordinary differential equations

$$(r\,R')' + \lambda^2\, r\, R = 0, \text{ with } R(a) = 0$$

and

$$T'' + \lambda^2 c^2\, T = 0.$$

We consider the equation in R first. This equation leads to the Bessel functions and, in view of the physical requirement that solutions stay bounded, we use only R(r) = BesselJ(λr). By now, you should know how to check to see that these functions really are solutions for the ordinary differential equation in R.

To meet the boundary condition, you recall that we use the zero's of the Bessel function. We prepare about 10 of them. Take a to be one for specificity.

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> a:=1;
> for n from 1 to 10 do
>     zero[n]:=evalf(BesselJZeros(0,n))/a;
> od;
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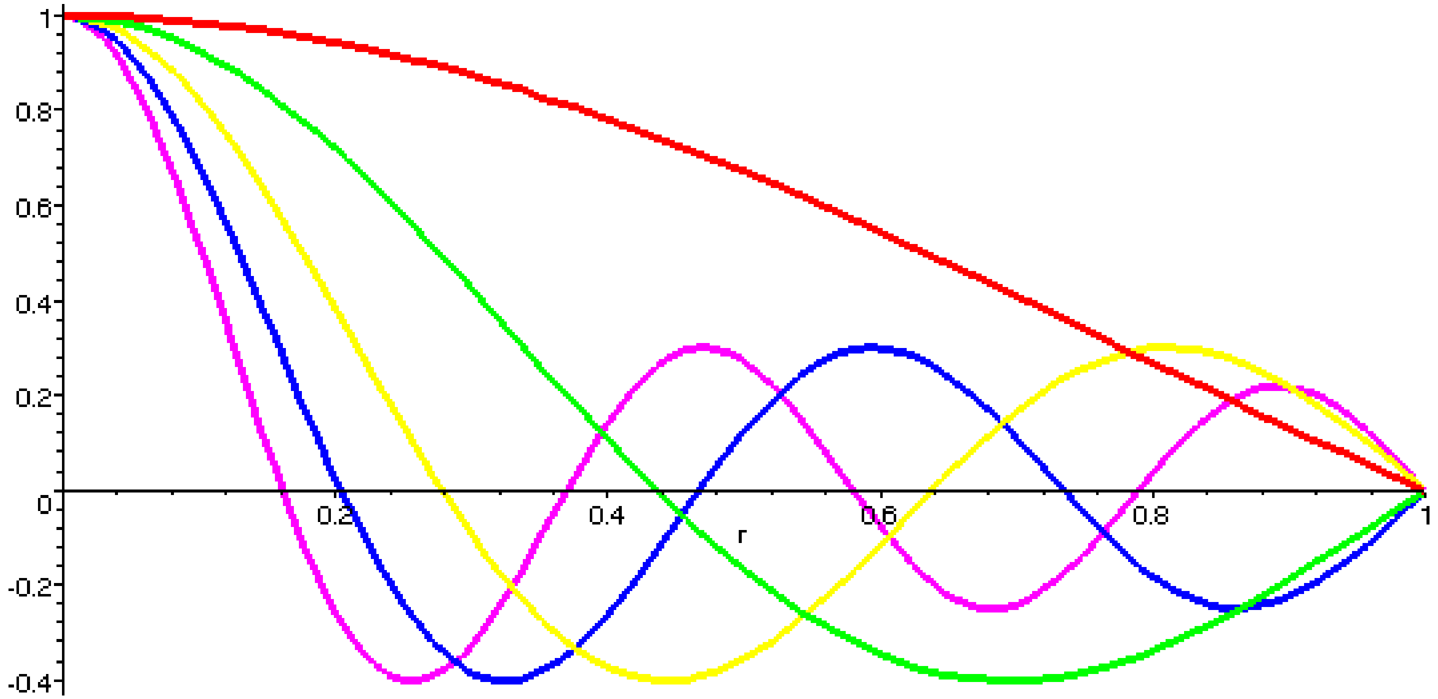
Define $R_n(r) = \text{BesselJ}(\text{zero}_n\, r)$

We draw graphs of the first five of these.



The equation in T is familiar. It leads to sines and cosines. Solutions for

$$T'' + \lambda^2 c^2 T = 0,$$

or, in this case,

$$T'' + zero_n^{\,2} c^2 T = 0,$$

are $a_n \cos(zero_n\ c\ t) + b_n \sin(zero_n\ c\ t)$.

Solutions for the partial differential equation are

$$u(t,r)=\sum_n R_n(r)\,[a_n \cos(\mathrm{zero}_n\ c\ t)+b_n \sin(\mathrm{zero}_n\ c\ t)].$$

Check these.

$$u(t, r) = \sum_n R_n(r)\,[a_n \cos(zero_n\ c\ t) + b_n \sin(zero_n\ c\ t)]$$

The initial condition asks that

$$f(r) = u(0, r) = \sum_n R(n,r)\, a_n$$

and that

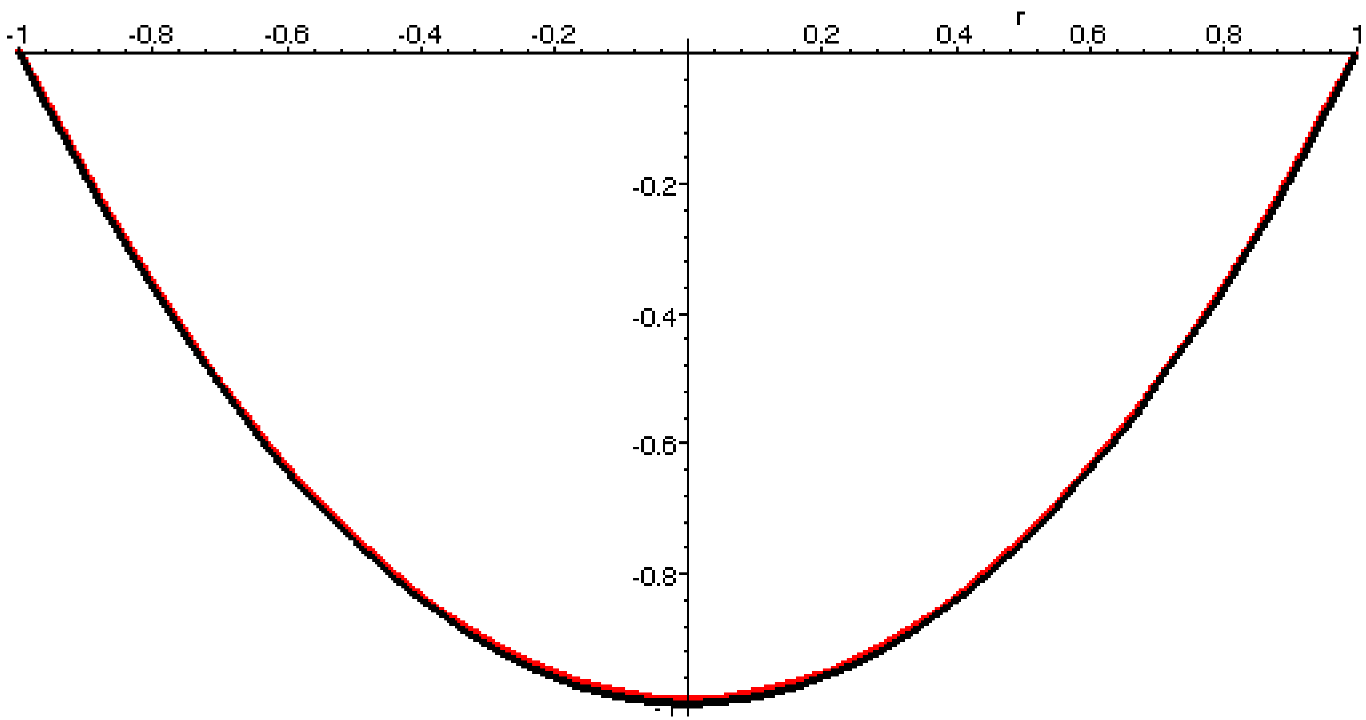
$$g(r) = \partial v/\partial(0, r) = \sum_n R(n,r)\, c\ zero_n\, b_n$$

We compute these as Fourier coefficients.

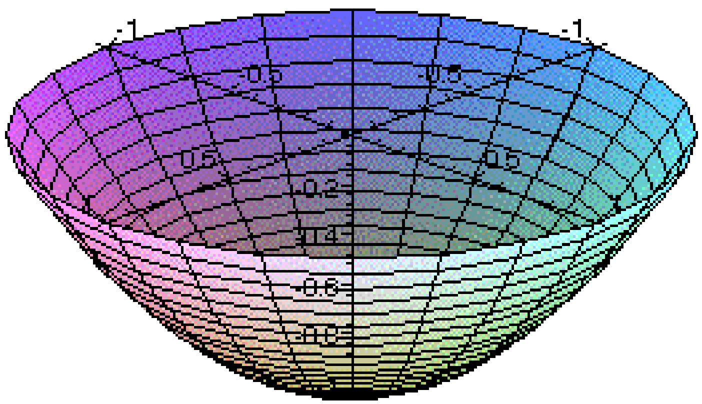
$$a_n = \int_0^1 f(r)\, R_n(r)\, r\, dr \,/ \int_0^1 R_n(r)^2\, r\, dr$$

$$zero_n\, b_n\, c = \int_0^1 g(r)\, R_n(r)\, r\, dr \,/ \int_0^1 R_n(r)^2\, r\, dr$$

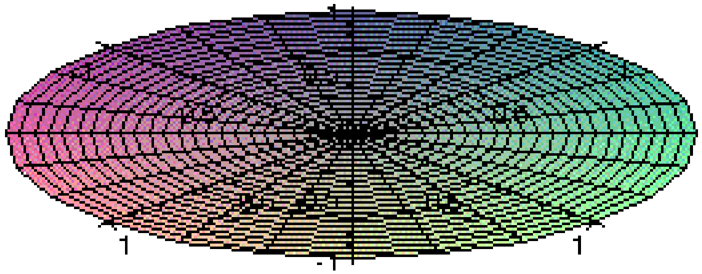
We do a specific example and watch the drum vibrate. We'll take $f(r) = r^2 - 1$ and $g(r) = 0$. For this example, take $c = 1$.



The initial condition:



Next, we take f(r) = 0 and g(r) not zero.



Assignment: See the MAPLE worksheet.

Summary: In this Module 36 we examined the vibrations of a circular membrane. The study took us to Bessel's functions.