Module 37: Laplace Transform in ODE's

Most students coming into these notes will have had an undergraduate exposure to ordinary differential equations and to the use of Laplace Transforms for solving these. In this first module on using Laplace transforms, we recall some of the elementary ideas associated with this subject.

We will exploit our advantage in having Maple to make computations.

Suppose that f(t) is sectionally continuous on every interval of the form [0, a]. The Laplace transform of f is defined as

$$L(f)(s) = \int_0^\infty \exp(-s\,t)\, f(t)\, dt$$

We give two simple examples: the Laplace transform of 1 and of exp(-3 t).

$$\int_0^\infty \exp(-s\,t)\, 1\, dt = 1/s \quad \text{and}$$

$$\int_0^\infty \exp(-s\,t)\, \exp(-3\,t)\, dt = 1/(s+3)$$

Of course, we do not have to evaluate integrals for every Laplace transform. Maple knows many Laplace transforms. We read in the transform package and obtain access to Laplace transforms.

```
> with(inttrans):
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To get the Laplace transform of f(t) going from the t variable to an s variable, type

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laplace(f(t), t, s).
```

Here are examples.

> with(inttrans):

> laplace(t,t,s); gives $1/s^2$

> laplace(t^2,t,s); gives $2/s^3$

> laplace(sin(a*t),t,s); gives $a/(s^2 + a^2)$

> laplace(cos(a*t),t,x); gives $s/(x^2 + a^2)$

Here is a guarantee that the Laplace transform can be computed: Suppose that f(t) is sectionally continuous for every interval of the form [0, a]. If, for some constant c, we have

$$\lim_{t \to \infty} [\exp(-c\ t)\ f(t)] = 0$$

then the Laplace transform of f exists for s > c. Such a function is said to be of exponential order.

What are some examples of functions of exponential order? Any bounded, sectionally continuous function has exponential order. Even functions of the form exp(a t) are. One function that is not of exponential order is $\exp(t^2)$,

Shifting Theorem: If L(F(t), t , s) = f(s), then

L(exp(a*t)*F(t), t, s) = f(s-a).

Shifting Theorem: If L(F(t), t , s) = f(s), then

L(exp(a*t)*F(t), t, s) = f(s-a).

Here's why:

$$f(s-a) = \int_0^\infty \exp(-(s-a)\,t)\, f(t)\, dt$$

$$= \int_0^\infty \exp(-s\,t)\,\exp(a\,t)\, f(t)\, dt$$

= L(exp(a*t)*F(t),t,s)

In using Laplace transforms to solve differential equations, it is important to be able to go backwards --- to go backwards in the sense that we can find f so that, for example,

$$L(f(t), t, s) = (s + 4)/(s^2 + 3 s + 2) .$$

Of course, Maple does this, too.

```
> invlaplace((s+4)/(s^2+3*s+2),s,t);
```

You may recall that humans typically found the inverse Laplace transform of the previous example by first performing a partial fraction decomposition.

```
> convert((s+4)/(s^2+3*s+2),parfrac,s);
```

gives $-2/(s+2) + 3/(s+1)$.

Then, a table look up for the inverse Laplace transform of 1/(s+2); and of 1/(s+1) , or

```
> map(invlaplace,[1/(s+2),1/(s+1)],s,t);
```

From the perspective of differential equations, the most important notion for Laplace transforms is what happens when we take the Laplace transform of y ' and of y ''.

How do humans think of this? Use integration by parts:

$$\int_0^\infty \exp(-(s-a)\, t)\, f'(t)\, dt = 0 - f(0) +$$

$$s \int_0^\infty \exp(-(s-a)\, t)\, f(t)\, dt.$$

A similar result holds for the Laplace transform of f ''.

We examine a few problems.

Problem 1:

We solve this differential equation two ways: using dsolve, and using laplace transforms.

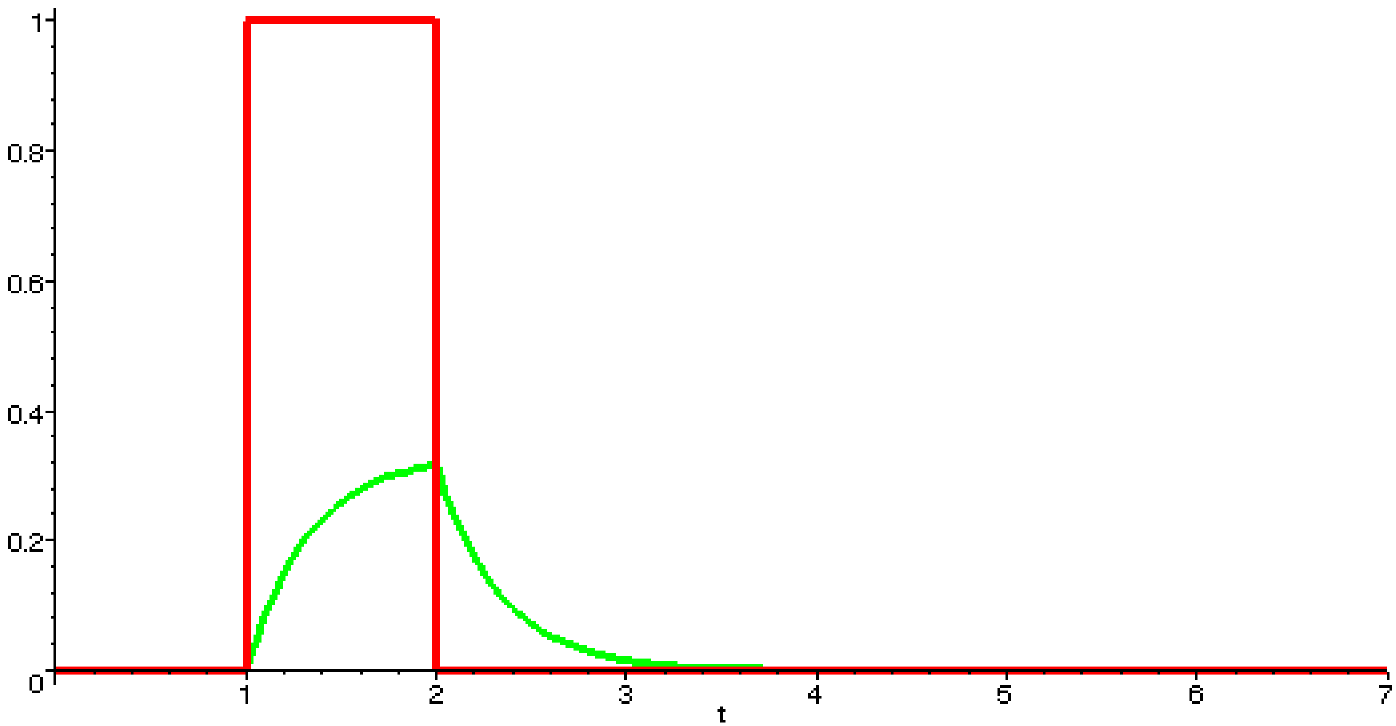
$$y'' + 3\,y' + 2y = 0,\; y(0) = 1,\; y'(0) = 0.$$

$$s^2\,h(s) - s\,1 - 0 + 3\,(s\,h(s) - 1) + 2\,h(s)=0.$$

Solve for h(s) and compute the inverse Laplace transform.

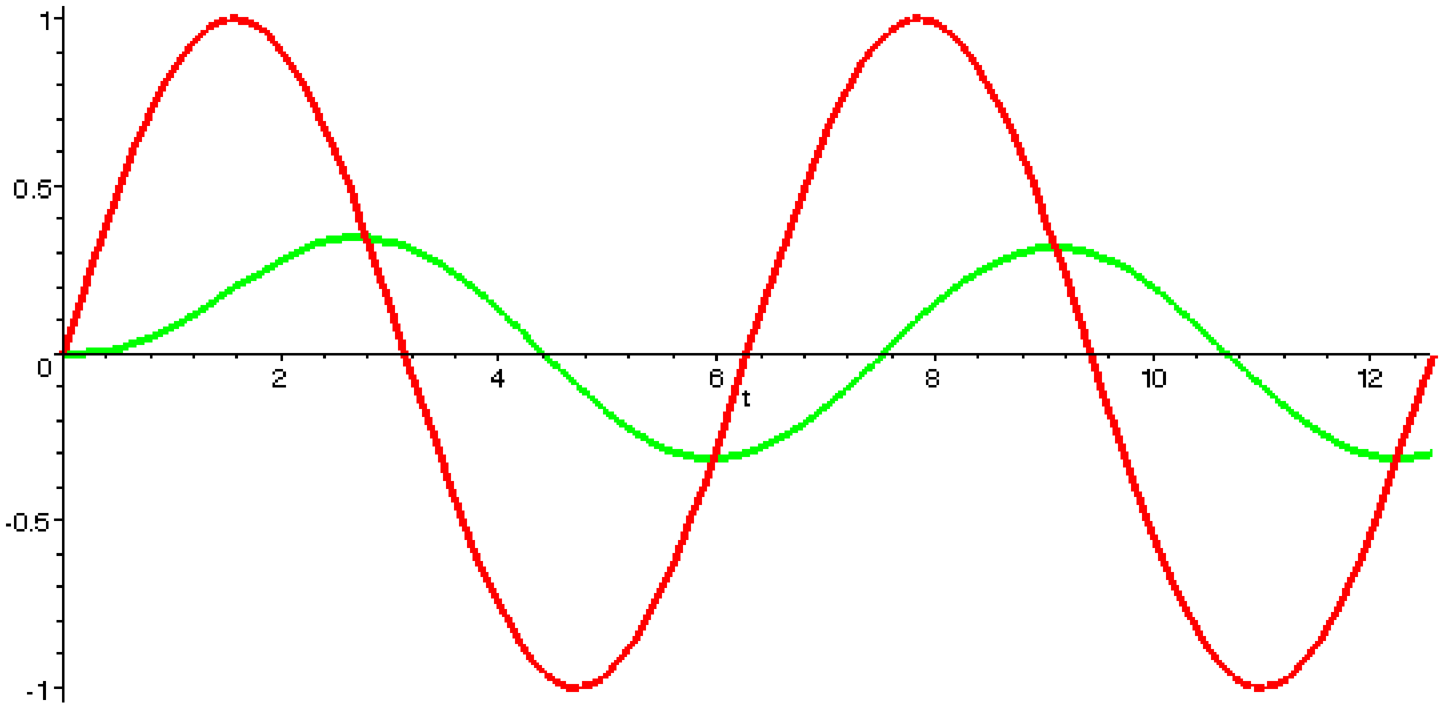
Problem 3: Graph the solution for

y ' + 3 y =Heaviside(t-1)-Heaviside(t-2), y(0)=0



Problem 4: Graph the solution for

y '' + 3 y ' + 2 y = sin(t), y(0) = 0 = y '(0).



Assignment: See Maple worksheet.

In this Module 37, we have

(1) given the definition for the Laplace Transform,

(2) presented the Shifting Theorem, and

(3) illustrated how the Laplace transform is used to solve ordinary differential equations.