Module 38: Transforms of the Heaviside and Dirac functions

Three functions which come up often are the Heaviside function, the Dirac function, and the Gamma function.

The Heaviside function is 1 for positive argument,
the Dirac function is a unit impulse, and
the Gamma function specializes as $\Gamma(n+1) = n!$.

The Heaviside Function

We have used the Heaviside function in work we have done previously. Recall the graph of

$$y(t) = \sin(\pi t)\ [\ \text{Heaviside}(t-2) - \text{Heaviside}(t - 4)]$$

Take the Laplace transform of this function

$$\int_0^\infty \exp(-s\ t)\ y(t)\ dt = \int_2^4 \exp(-s\ t)\ \sin(\pi t)\ dt\ .$$

The Dirac Delta function.

A different function is this Dirac Delta function.

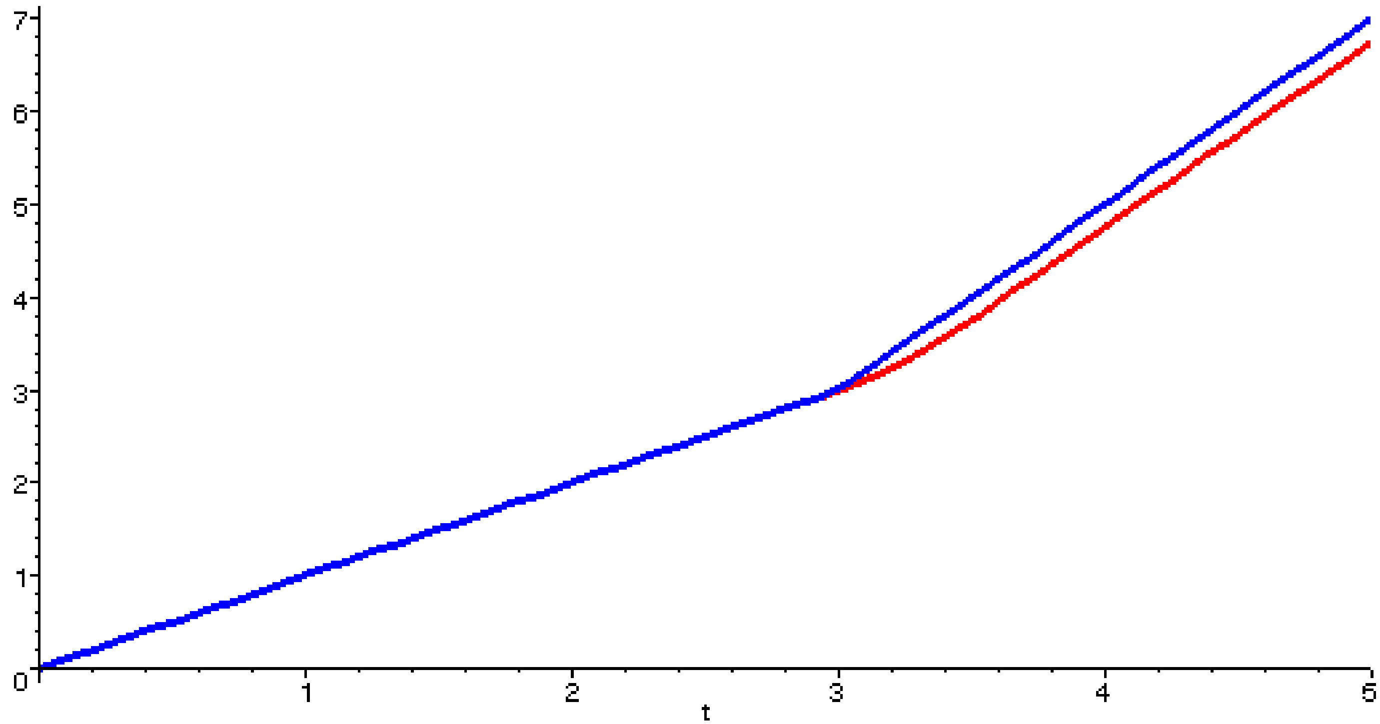
Consider the forcing function
f(t) = (Heaviside(t-3)-Heaviside(t-3-h))/h,
where h > 0.
We ask what is the solution for the differential equation

$$y''(t) = f(t), \quad y(0) = 0, \; y'(0) = 1.$$

$f(t) = (\text{Heaviside}(t-3)-\text{Heaviside}(t-3-h))/h,$

$y''(t) = f(t), \; y(0) = 0, \; y'(0) = 1.$

Graph $y(t, h)$ and $\lim_{h\to 0} y(t, h)$.



This last suggests what is true.

$\lim_{h->0}$ (Heaviside(t-3)-Heaviside(t-3-h))/h =

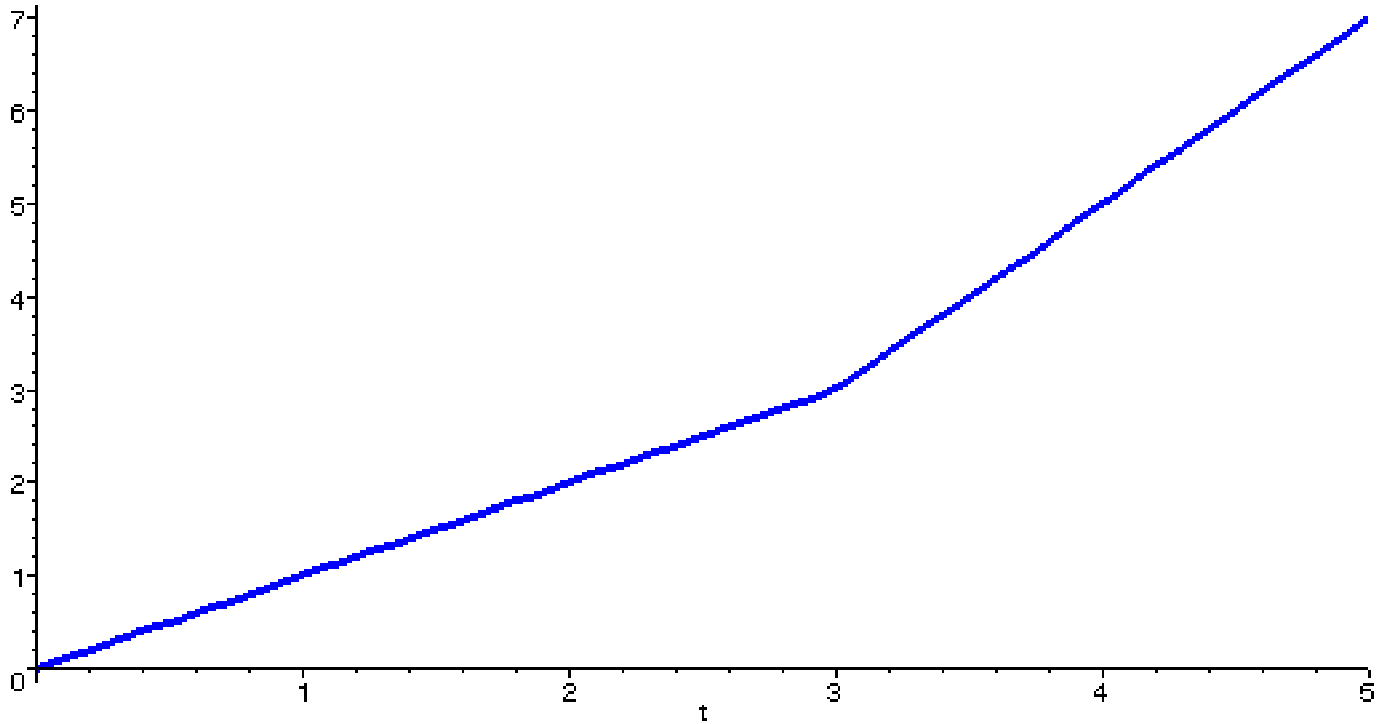
Dirac(t-3).

Or, what is the same,
d Heaviside(t-3)/dt = Dirac(t-3).

We can take the Laplace transform of the Dirac function, and solve differential equations with it as a forcing function.

The Laplace transform of Dirac(t-3) is

$$\int_0^\infty \exp(-s\,t)\,\mathrm{Dirac}(t-3)\,dt = \exp(-3\,s).$$



There are two theorems that could be called Translation Theorems or Shifting Theorems. The first has been discussed.

Translation Theorem 1: If L(F(t),t,s)=f(t), then L(exp(a t) F(t), t,s)=f(s-a).

Reference: Module 37. The following example is an illustration.

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> laplace(t^2,t,s);
> laplace(exp(-3*t)*t^2,t,s);
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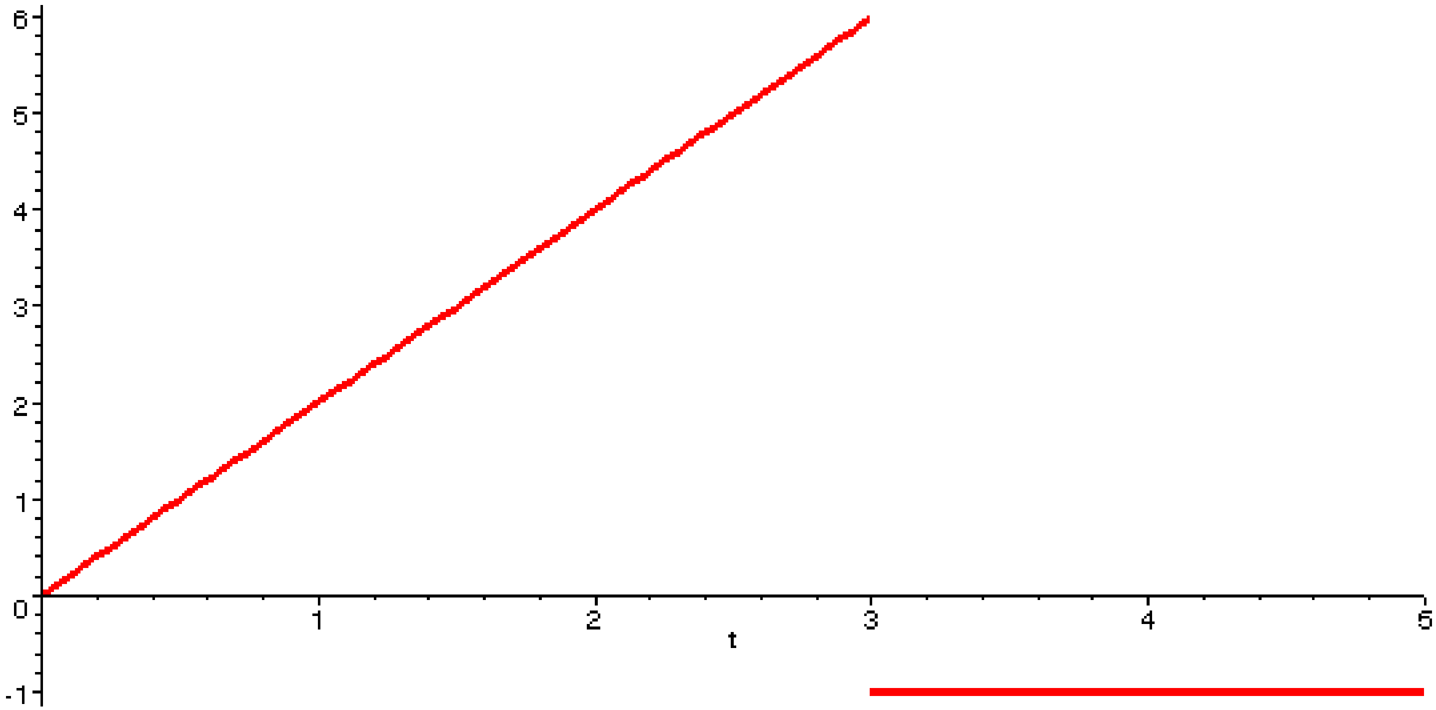
Make up a sum of Heaviside functions to produce this graph.

Ans: f(t) = 2 t - (2 t+1) Heaviside(t -3)



How you would use this result to compute the inverse Laplace transform for

$$3/(s^2 - 8 s + 25) = 3/((s-4)^2+9)?$$

Translation Theorem 2: If Laplace(F(t), t, s) = f(s), then for any positive constant b, then

Laplace(Heaviside(t-b) F(t-b),t,s) = exp(-b s) f(s).

A proof of this Translation Theorem 2.

We suppose that

$$f(s) = \int_0^\infty \exp(-s\ t)\ F(t)\ dt$$

Then,

$$\exp(-b\ s)\ f(s) = \int_0^\infty \exp(-s\ (t+b))\ F(t)\ dt$$

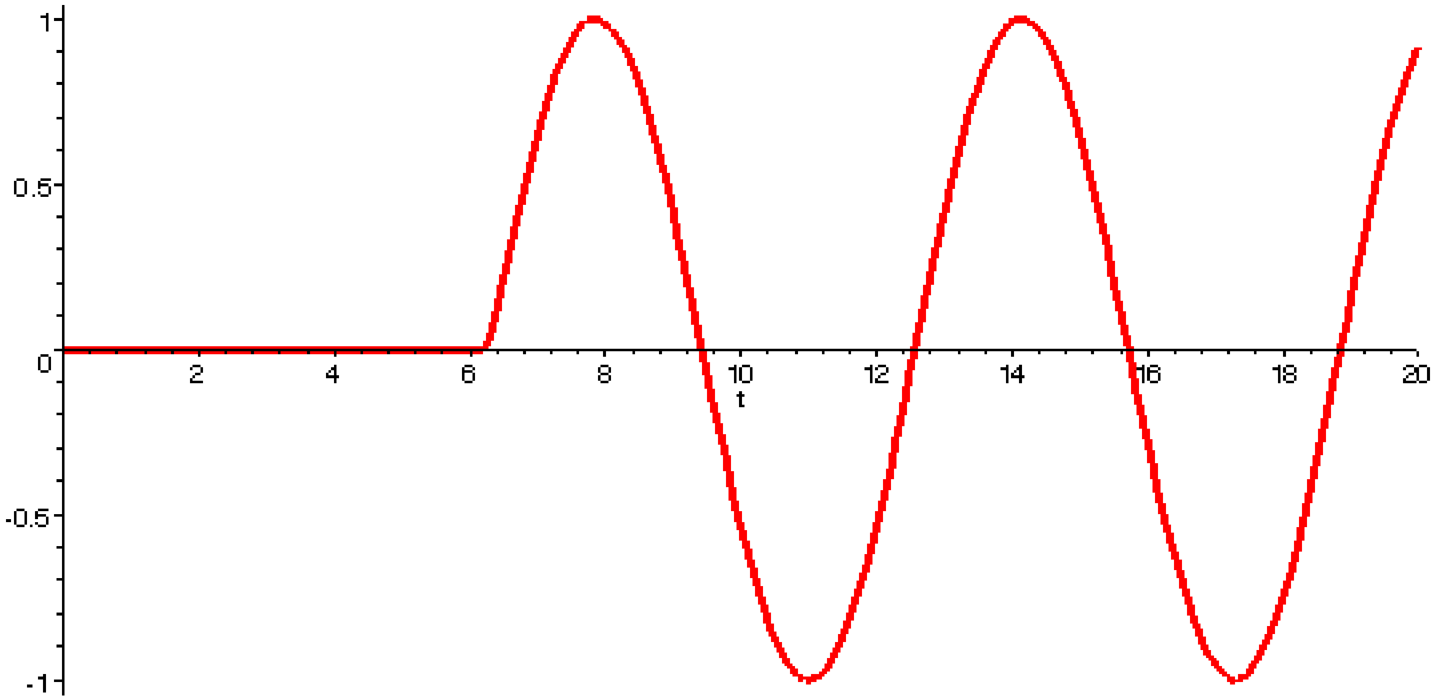
$$\exp(-b\ s)\ f(s) = \int_0^\infty \exp(-s\ (t+b))\ F(t)\ dt$$

Do a change of variable letting $\tau = t+b$.

$$\exp(-b*s)*f(s) = \int_b^\infty \exp(-s\ \tau\ )\ F(\tau-b)\ d\tau$$

= Laplace(Heaviside(t-b) F(t - b), t, s).

Here is the graph of a function for which we want to find the Laplace transform. How is this done?



Using the fact that sin(t) is periodic with period 2 π, we can write

sin(t) Heaviside(t - 2 π)

as

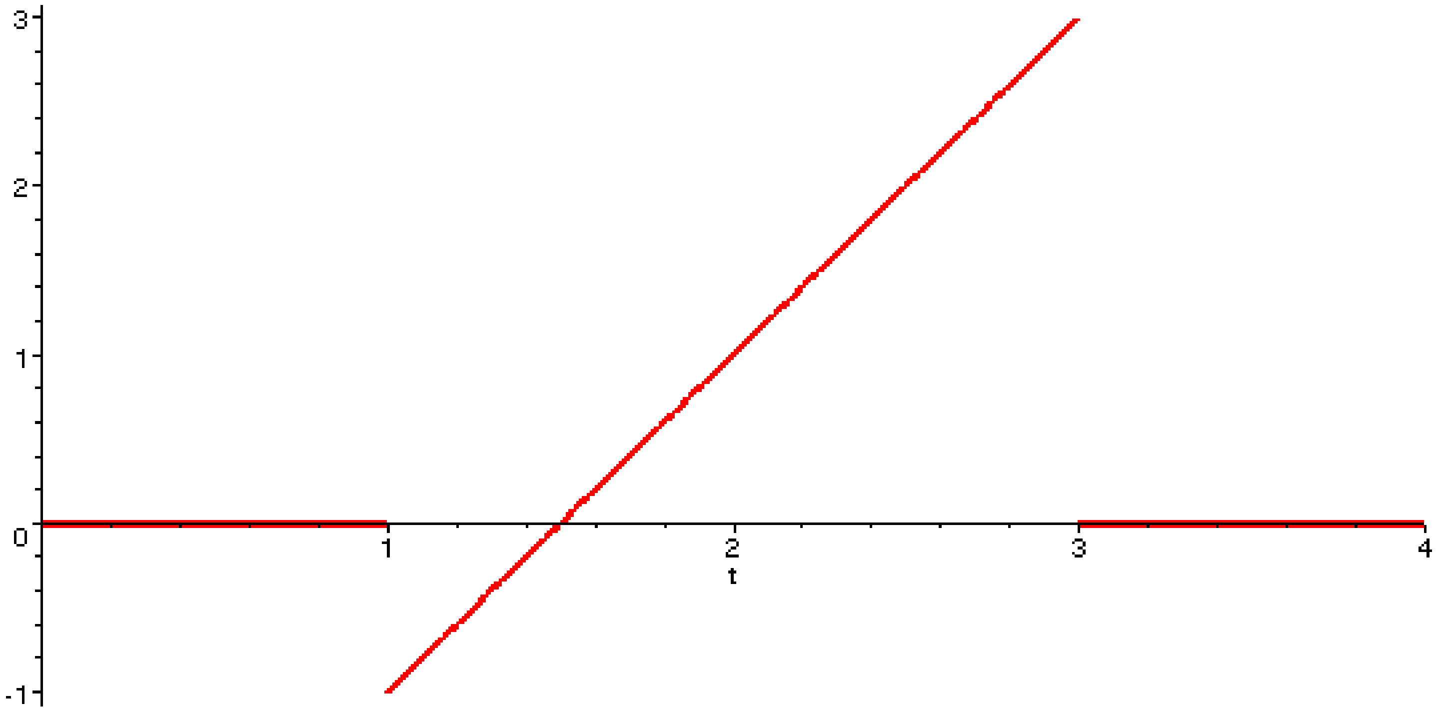
sin(t -2 π) Heaviside(t - 2 π).

Consequently,

Laplace(sin(t) Heaviside(t - 2 π), t, s) =

exp(- 2 π s) Laplace(sin(t), t, s).

The following is an exercise in translating a graphical picture of a function to a function with which we can compute the Laplace transform.



The Gamma function: Gamma(x).

The Laplace transform of a power function t^n is often expressed in terms for the gamma function, which is defined in terms of an integral:

$$\mathrm{Gamma}(x) = \int_b^\infty \exp(-t)\, t^{x-1}\, dt$$

First we see that Gamma(1) = 1.
Gamma(x+1) = x Gamma(x), use integration by parts.

We illustrate the techniques for asking Maple to use Laplace transforms by giving a differential equation for which Maple needs the Laplace techniques. The equation is

$$dy(t)/dt + y(t) = 1 - \int_0^t y(s)\, ds, \text{ with } y(0) = 0.$$

Assignment: See Maple Worksheet.

In this Module 38, we have

(1) computed the Laplace transform of the Heaviside and Dirac functions, and

(2) presented the translation theorems for Laplace transforms.