

Module 39: Periodic Forcing Functions

For a variety of reasons, periodic functions arise in natural phenomena, either as forcing functions for systems or as states of systems. They arise so often that their appearance should be anticipated.

Of course, if you know a periodic function on one period, in a very real sense, you know it everywhere. It is no surprise that the Laplace transform can be computed by integrating over only one period.

Here is the Theorem:

If F is periodic with period T and sectionally continuous and if $f(s) = L(F(t), t, s)$

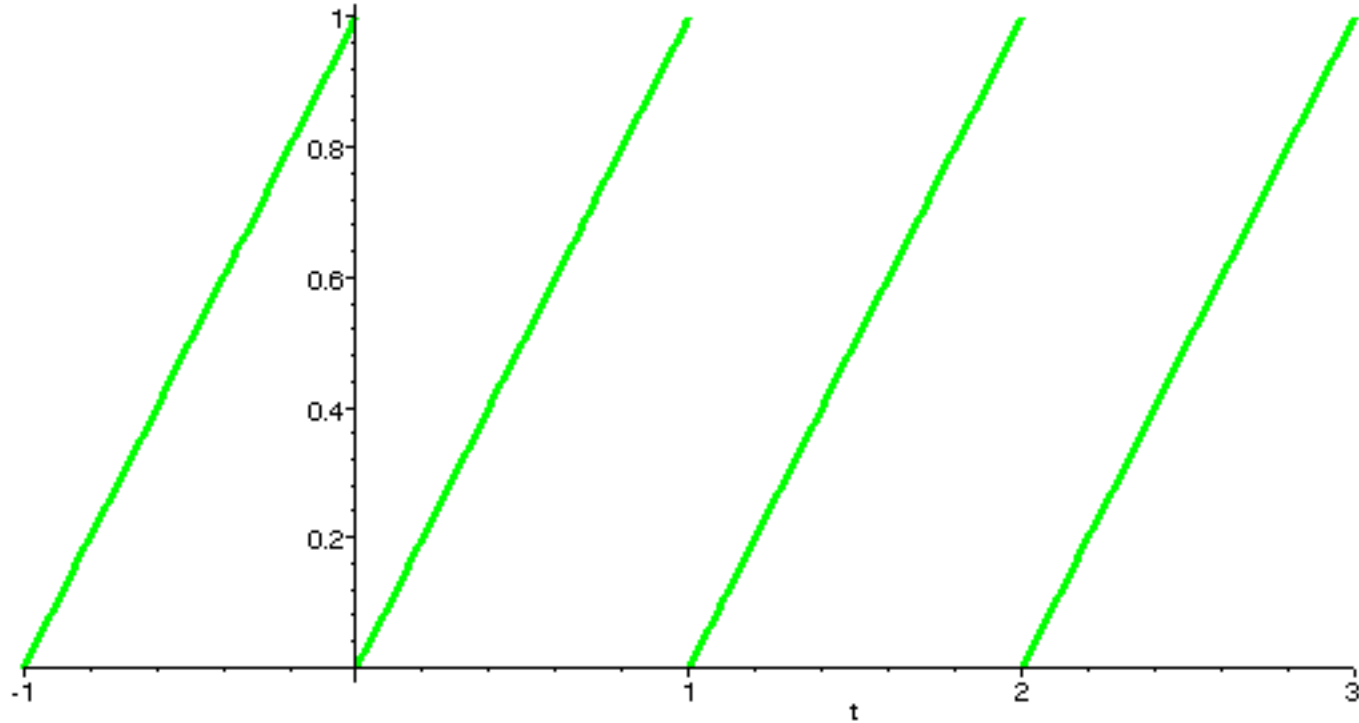
then

$$f(s) = \int_0^T \exp(-s t) F(t) dt / (1 - \exp(-s T)).$$

Proofs for this can be found in standard texts.

We present an example.

Consider the function with period 1 that is t on the interval from 0 to 1. We graph this function.

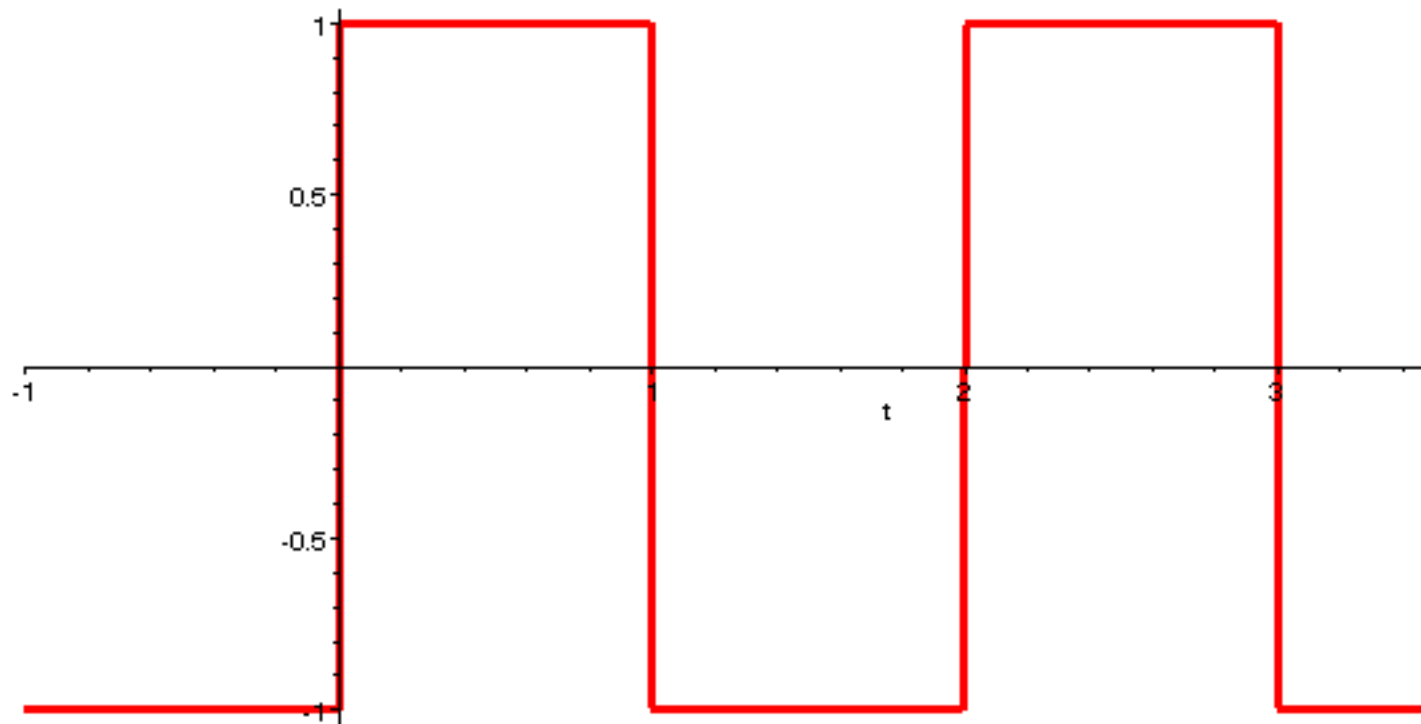


Example 1. $t - \text{floor}(t)$

$$f(s) = \int_0^1 \exp(-s t) t \, dt / (1 - \exp(-s))$$
$$= (s + 1 - \exp(s)) / (s^2 (1 - \exp(s))).$$

Another example would be a function which has period 2, which is -1 on $[-1, 0]$ and which is 1 on $[0, 1]$.

Example 2: Odd extension of Heaviside(t) with period 2. This is the signum function of period 2 as extended to the positive numbers.



The result will be

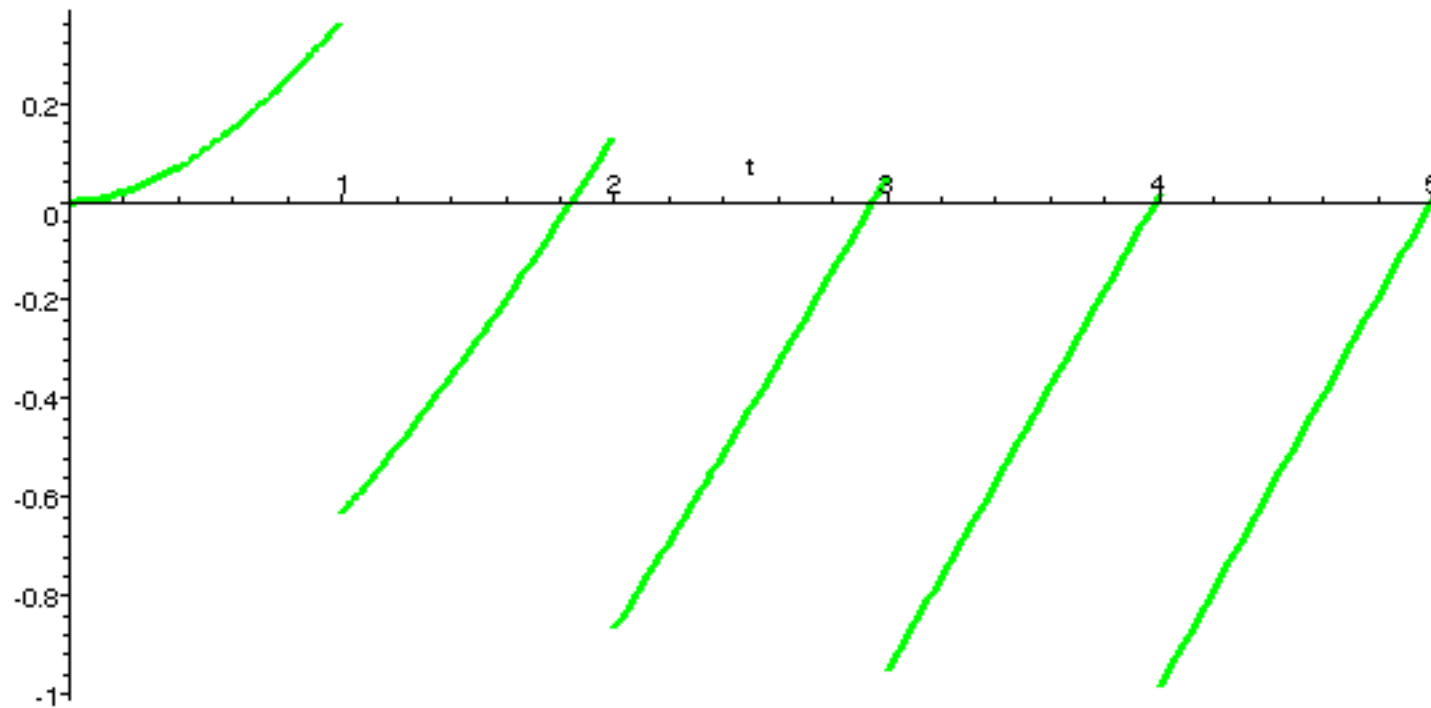
Laplace of the odd, periodic extension of Heaviside(t):

$$= \left[\int_0^1 \exp(-s t) dt - \int_1^2 \exp(-s t) dt \right] / (1 - \exp(-2 s))$$

The result is $(1 - \exp(-s)) / (s (1 + \exp(-s)))$.

Consider $(-1)^{\text{floor}(t)}$.

We turn again to the equation $y' + y(t) = t - \text{floor}(t)$, $y(0) = 0$.

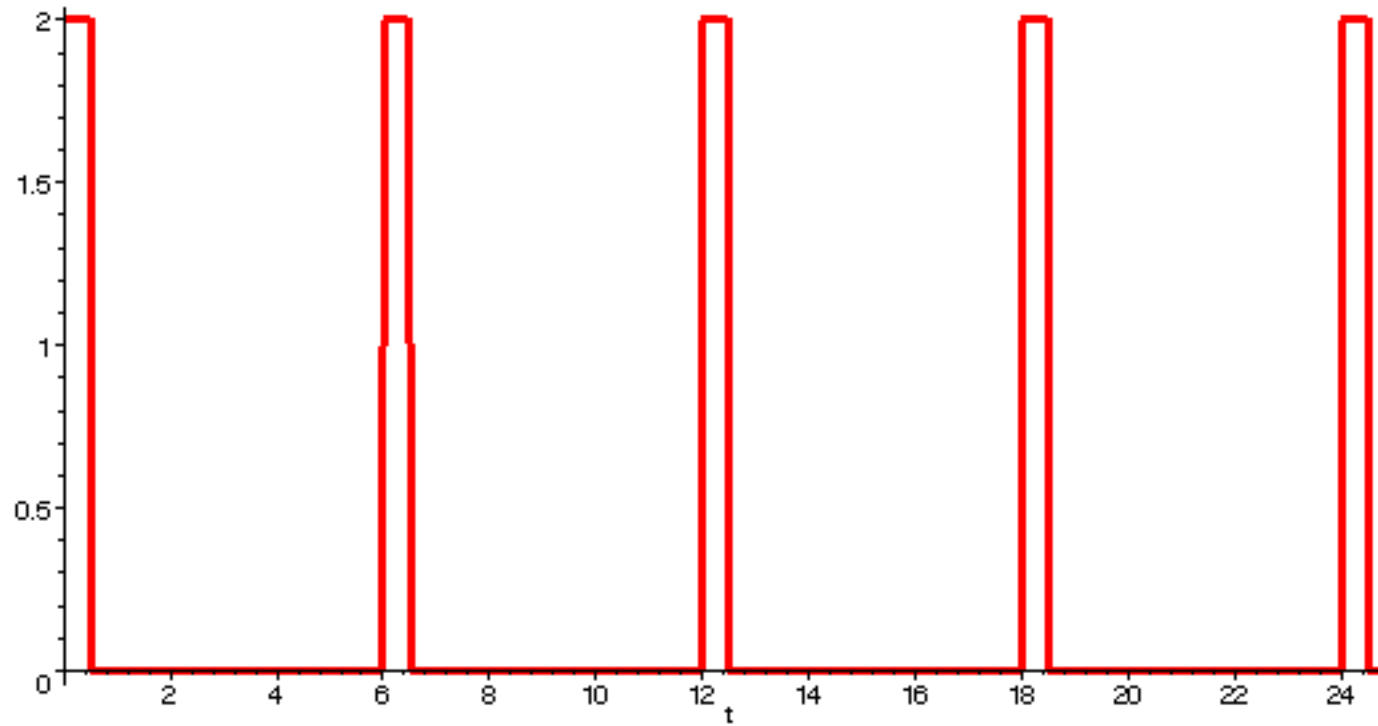


An Application from Pharmacology

A drug is taken every six hours. We keep track of the concentration of the drug in the GI tract and in the circulatory system.

We think of the dosage as follows: Think of one drug cycle as a six hour block. Suppose the treatment begins at time 0. Every six hours, another dosage is taken. During the first half hour of that time the dosage comes in at the rate of 2 units per hour, so that a total of 1 unit is received. Here is our model of that dosage.

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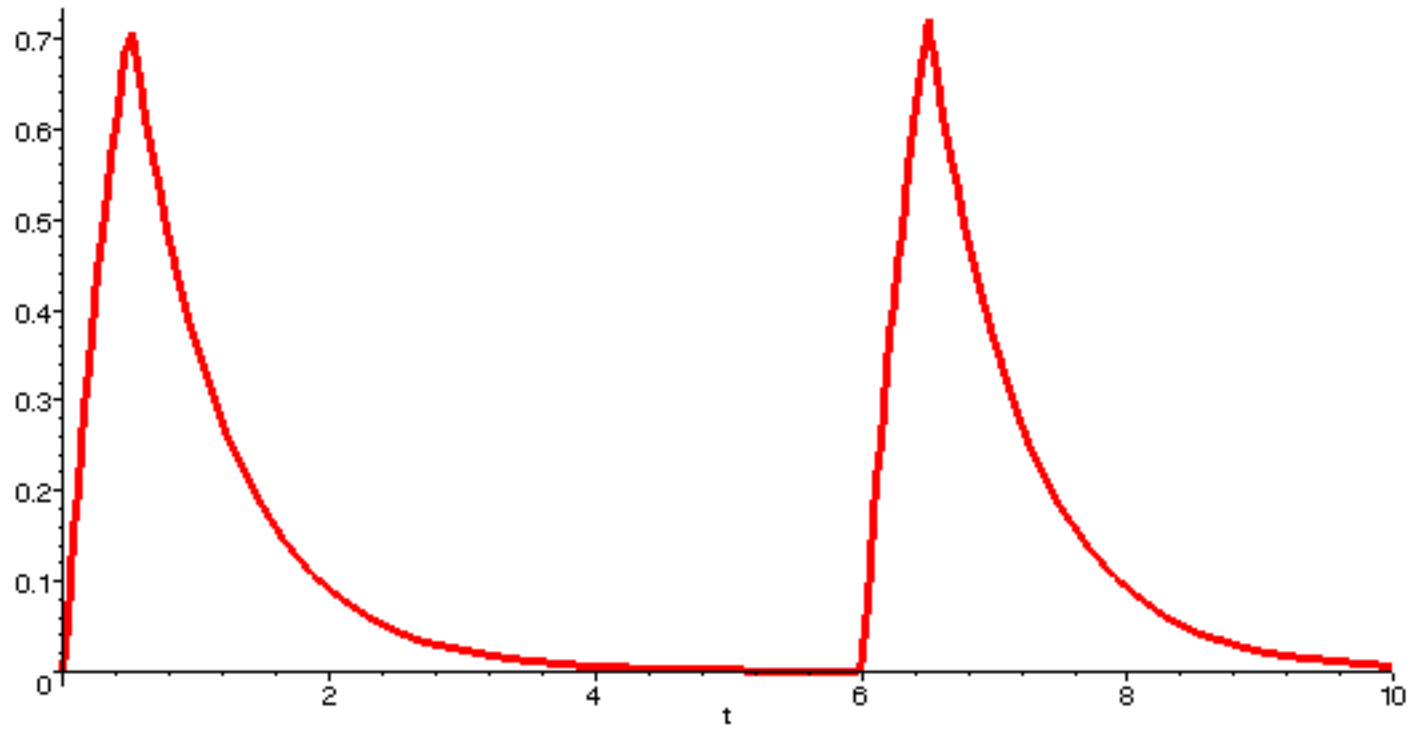
We describe how the drug moves from the GI tract to the circulatory system analytically.

The drug is removed from the GI tract at a rate proportional to the amount present.

$$d x(t)/dt = - a x(t)+D(t) \text{ with } x(0) = 0.$$

In the Maple worksheet I suggest methods for solving this equation by numerical methods.

Solution



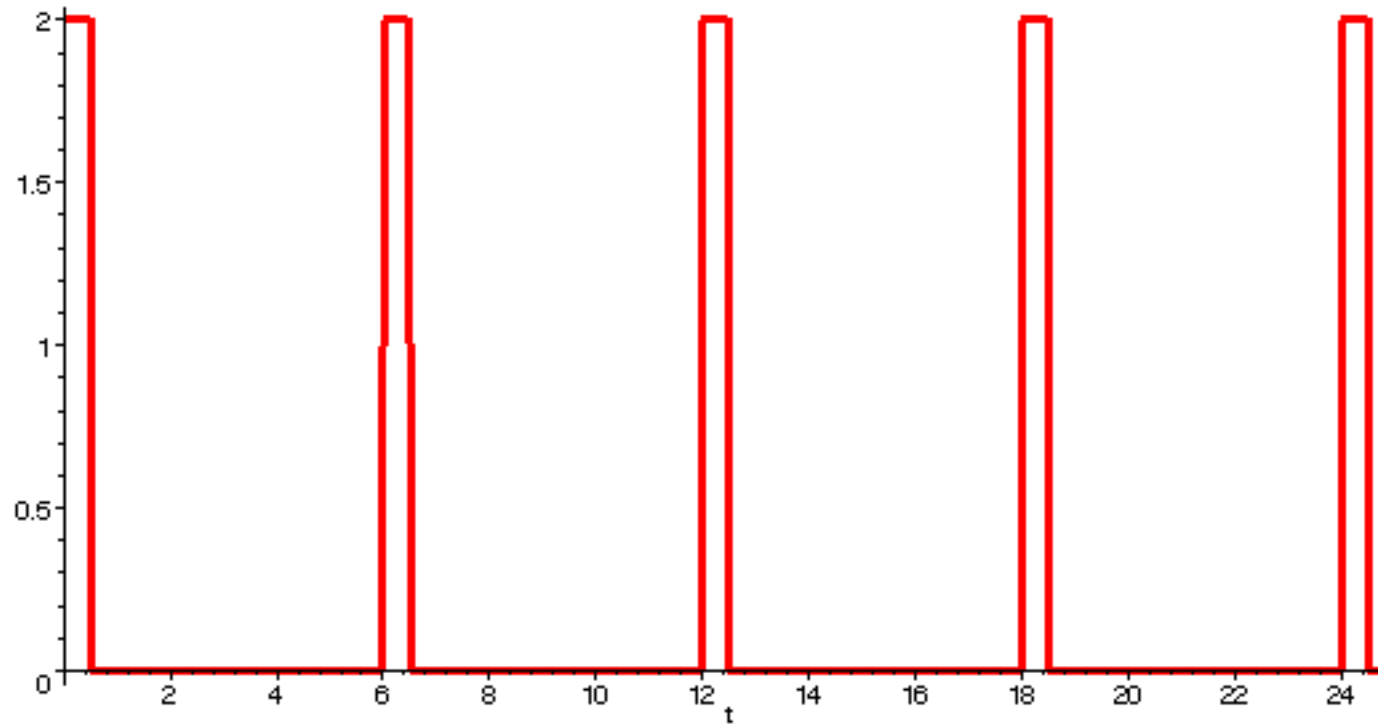
Analytic Methods

We use the Heaviside function to rewrite Dose so that we can use analytic methods for solving this equation.

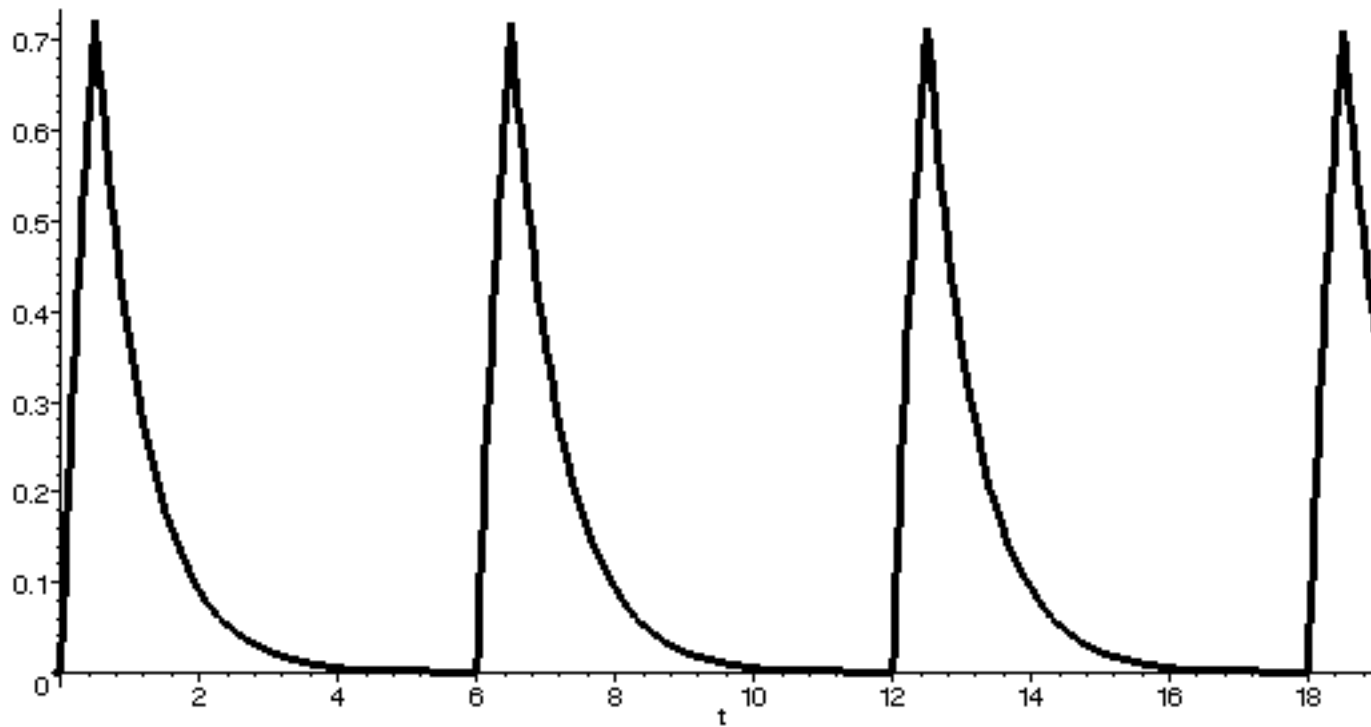
$$\text{Dose}(t) = \frac{2}{p} (\text{Heaviside}(t-6p) - \text{Heaviside}(t-(6p+1/2)))$$

Look carefully at the graph.

Here is our model of that dosage.



Now solve the equation.

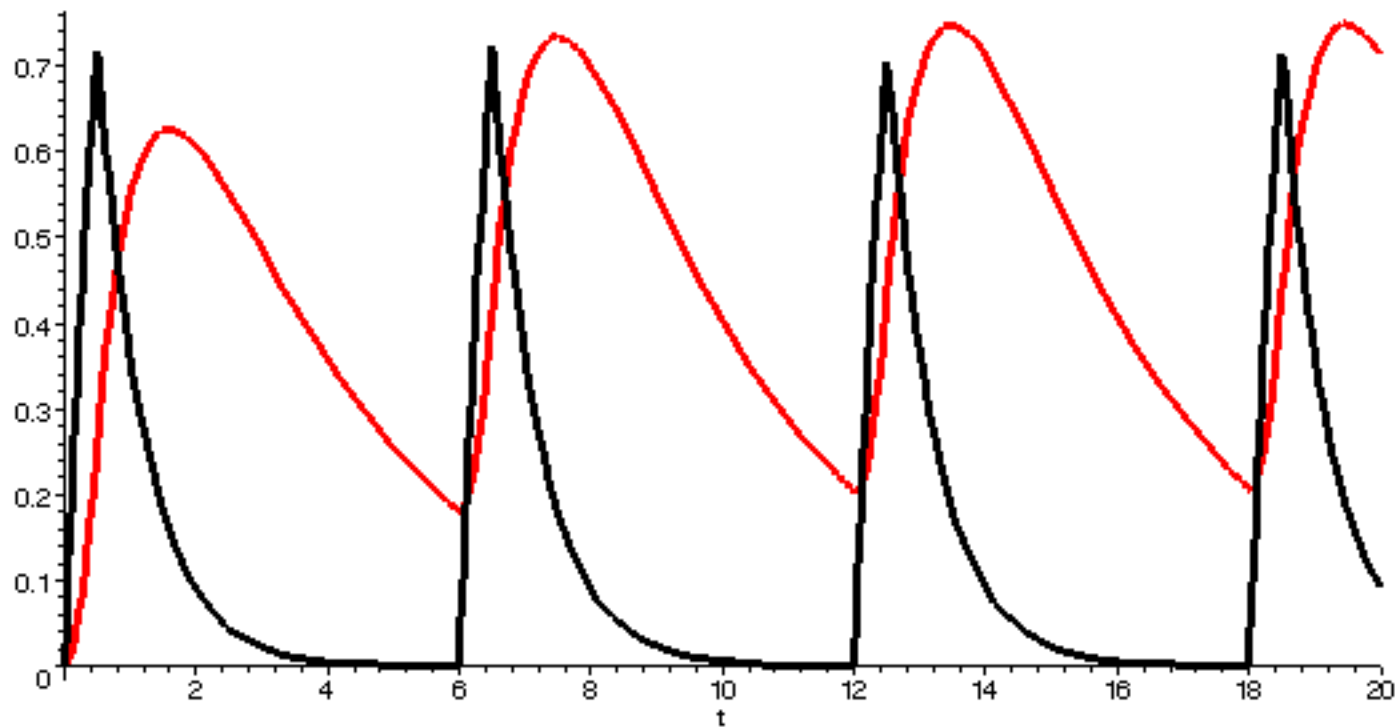


We add one more idea. Let the drug move from the GI tract into the circulatory system.

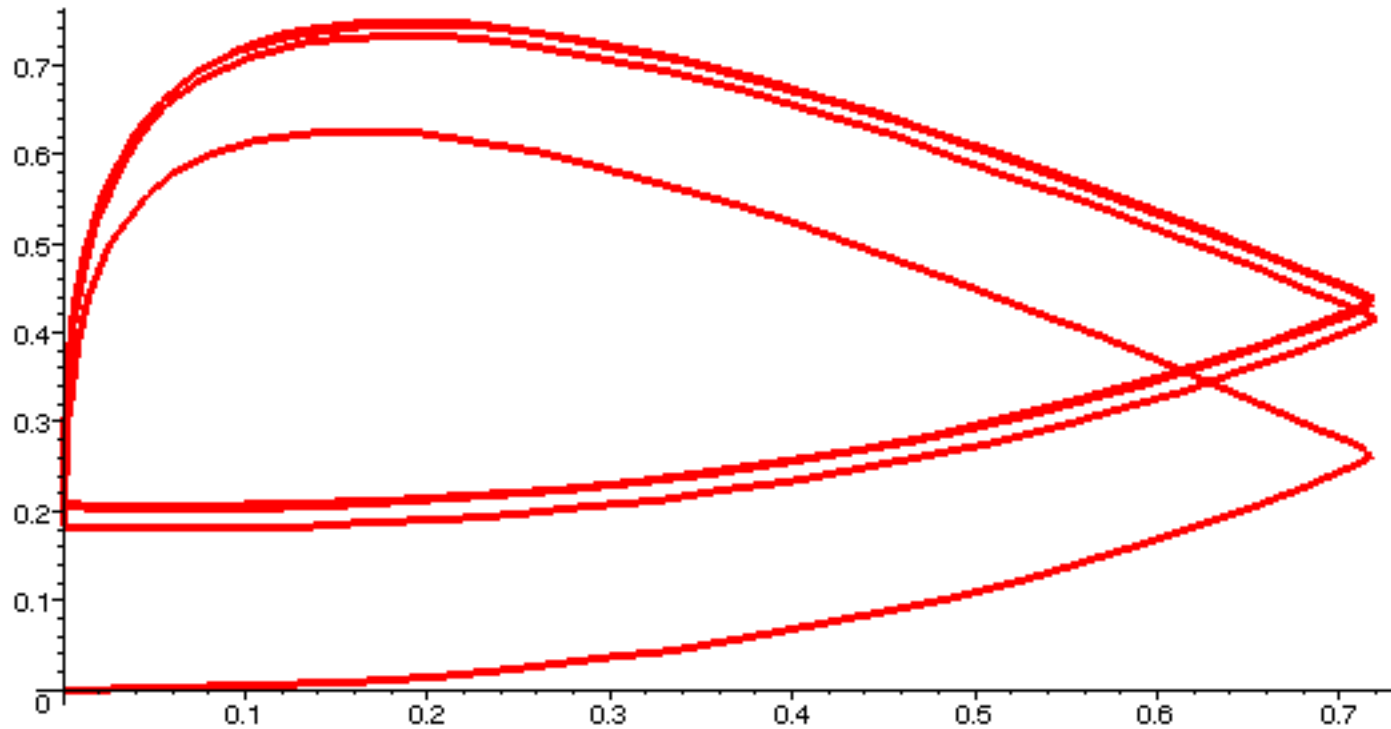
There is more to using the drug than having it move into the GI tract. It also is metabolized from the blood stream. The amount of the concentration in the circulatory system decreases proportioned to the amount present. Of course, the concentration increases with movement of the drug from the GI tract.

Thus, we have the second equation to form a system:

$$dy(t)/dt = a x(t) - b y(t).$$



Here is a graph of the solutions plotted parametrically.



Periodic solutions

Looking at this graph gives indication that there is likely a periodic solution for the system. We seek that solution. For such a solution, it will be true that when $t = 6$ we are back at the same place: that is, $x(0) = x(6)$ and $y(0) = y(6)$. We seek such a value for $x(0)$ and $y(0)$. The task at hand is to find an initial condition that will also be the value $t = 6$.

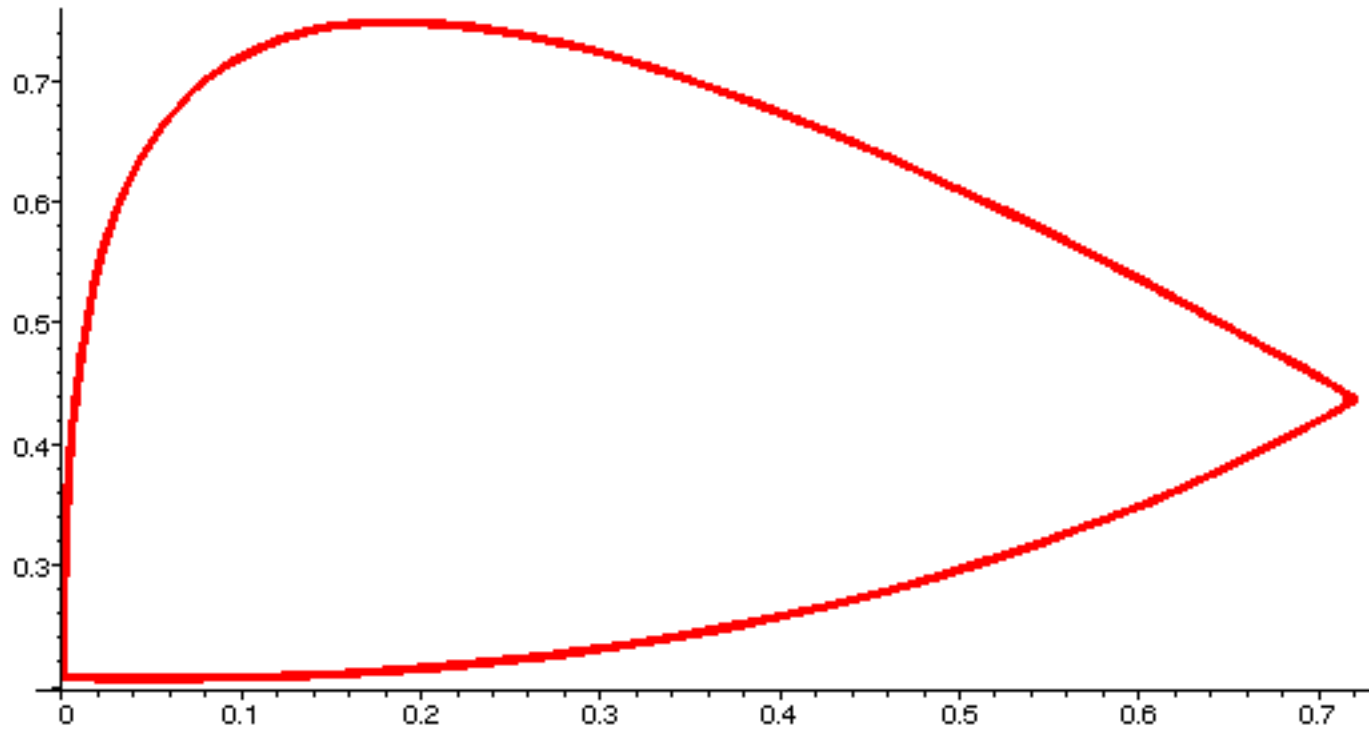
Find the initial condition. Solve

$$\begin{aligned}x' &= -a x(t) + \text{dose}(t), \quad x(0) = c, \\y' &= a x(t) - b y(t), \quad y(0) = d.\end{aligned}$$

Then, find c and d so that

$$x(0) = x(6) \text{ and } y(0) = y(6).$$

Graph for the periodic solution.



A Mathematical Principle

This previous calculation suggests a mathematical principle:

THEOREM: Suppose that c is not zero and that f is a piecewise continuous, periodic function.

There is a number y_0 such that the solution for

$$y'(t) + c y(t) = f(t) \quad \text{with} \quad y(0) = y_0$$

is periodic. Moreover, if z is another solution of this equation starting at z_0 , then

$$|y(t) - z(t)| = \exp(-c t) |y_0 - z_0|.$$

Assignment: See the Maple worksheet.

In this Module 39, we have

- (1) found methods for computing the Laplace transform of periodic functions,
- (2) solved differential equations with periodic forcing functions, and
- (3) found periodic solutions for differential equations which have periodic forcing functions.