Module 40: First Order Partial Differential Equations.

The power of Laplace Transforms is that they change partial differential equations to ordinary differential equations and ordinary differential equations to algebraic equations. Of course, there is always the problem of computing inverse Laplace Transforms. This may need to be done twice. We illustrate these techniques in this worksheet.

For emphasis, we illustrate these ideas with an ordinary differential equation. We have solved such equations before.

$$y'' + 6y' + 34y(t) = 30\sin(2t),$$

with $$y(0) = y'(0) = 0.$$

This problem models a mass attached to a spring with a periodic forcing function. We expect the system to begin from rest, but to tend toward a periodic oscillation. The peaks for the solution should follow those of the periodic forcing function.

Here are the details:

The ODE: $y'' + 6y' + 34y(t) = 30\sin(2t)$,

with $y(0) = y'(0) = 0$.

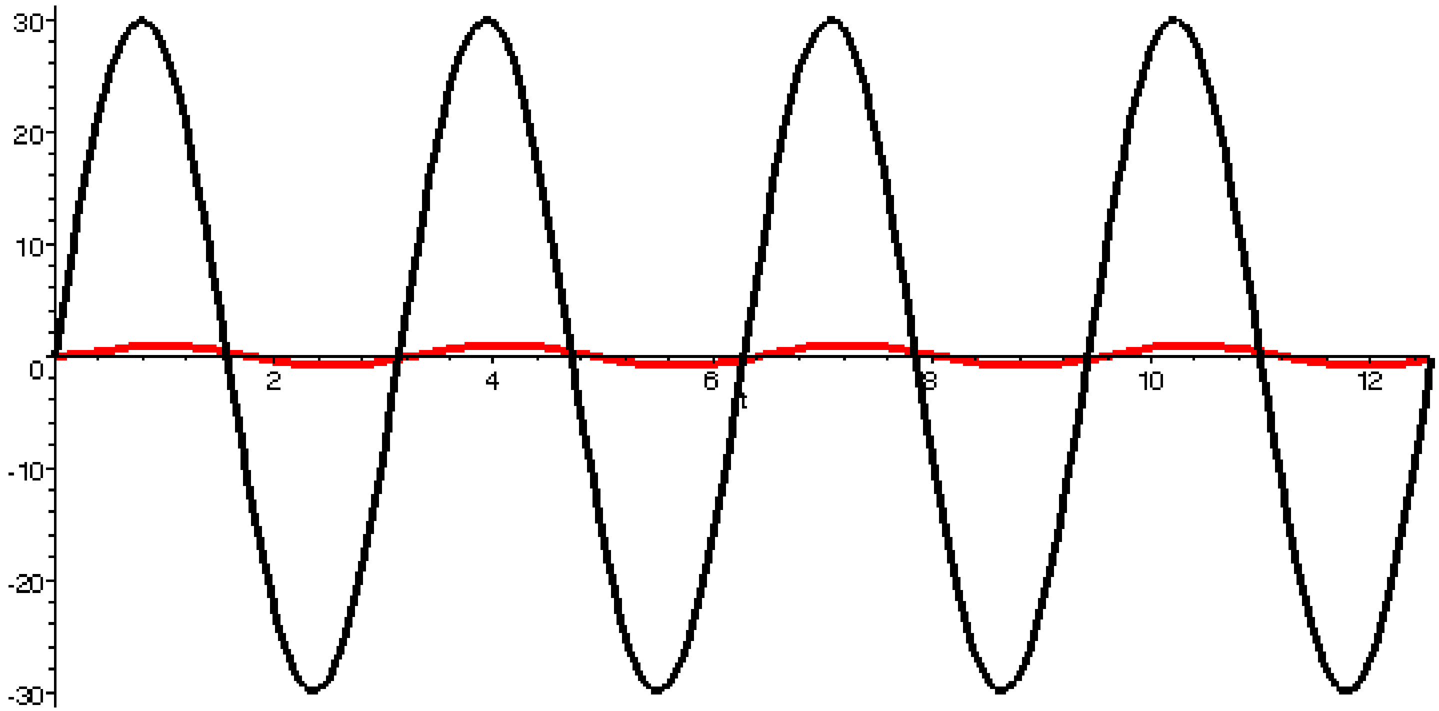
The Laplace Transform of the ODE:

$(s^2 + 6s + 34)\,\text{laplace}(y)(s) = 60/(s^2 + 4)$

$\text{Invlaplace}(60/[(s^2 + 4)(s^2 + 6s + 34)])(t)$

provides the solution.

Solution for the ODE.



We now solve first order partial differential equations. The pattern which was set in the previous example persist. These first order partial differential equations do not fit the pattern of separation of variables which we have used before. The methods of Laplace Transforms certainly provide one way to think of solving these first order systems. (There are other methods.)

We begin by trying four simple examples.

Here is the first example: $t > 0$, $x > 0$

$$u_t + u_x = 0,\ u(0,x) = \sin(x),\ u(t,0) = 0.$$

This problem asks what a surface should look like whose domain is in the first octant and for which the graph satisfies these conditions above. We answer this question in the context of this course by using the methods of Laplace Transforms.

PDE1:

$$u_t + u_x = 0,$$
$$u(0, x) = \sin(x),\ u(t, 0) = 0.$$

With $v(x) = \text{laplace}(u(t, x), t, s)$,

we have

$$s\, v(x) - u(0,x) + v\,'(x) = 0,$$

or $$s\, v(x) - \sin(x) + v\,'(x) = 0.$$

This is an ODE.

ODE: $s\, v(x) - \sin(x) + v'(x) = 0,\ v(0) = 0.$

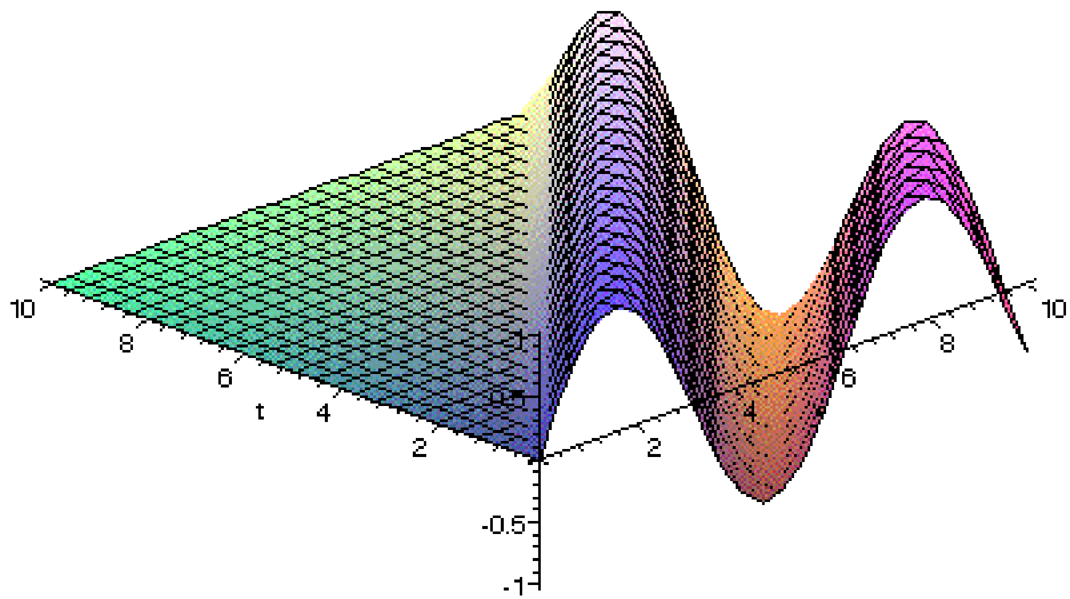
has solution

$[s \sin(x) - \cos(x) + \exp(-s\, x)] / (s^2 + 1).$

This has inverse laplace transform

$\sin(x - t) - \text{Heaviside}(t - x) \sin(x - t)$

Solution for PDE1



PDE2:

$$u_t + u_x = - u(t, x)$$

$$u(0, x) = \sin(x), \quad u(t, 0) = 0.$$

With,

$$v(x) = \text{laplace}(u(t, x), t, s),$$

we have

$$s\, v(x) - u(0,x) + v\,'(x) = - v(x),$$

or

$$s\, v(x) - \sin(x) + v\,'(x) = - v(x).$$

This is an ODE.

ODE: $s\,v(x) - \sin(x) + v'(x) = -v(x),\ v(0) = 0,$

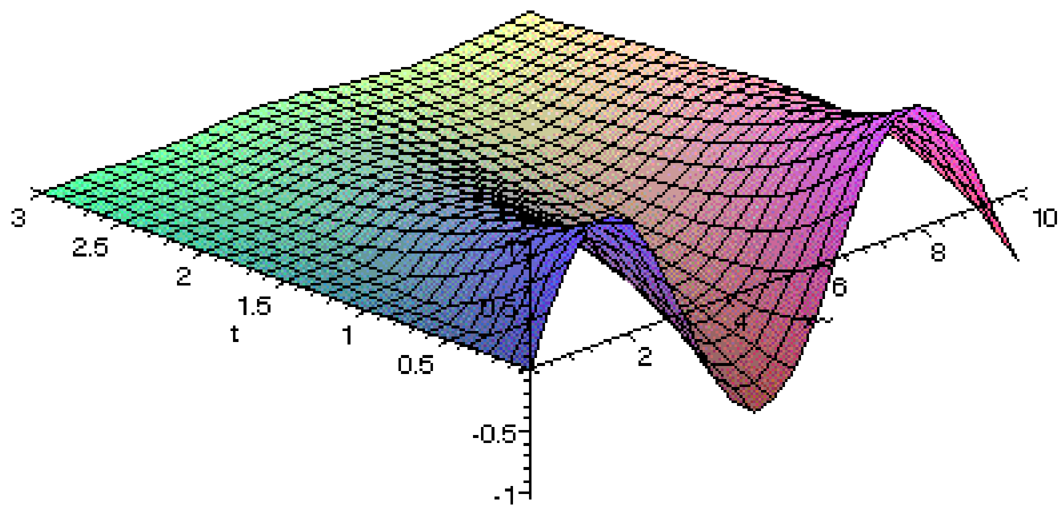
has solution

$(s+1)\sin(x) - \cos(x) + \exp(-(s+1)\,x)\ ]\ /\ (s^2+2s+2).$

This has inverse laplace transform

$[\ \sin(x - t) - \text{Heaviside}(t - x)\sin(x - t)\ ]\ \exp(-t).$

Solution for PDE2



PDE3:

$$u_t + u_x = + u(t, x)$$

$$u(0, x) = \sin(x), u(t, 0) = 0.$$

With,

$$v(x) = \text{laplace}(u(t, x), t, s),$$

we have

$$s\, v(x) - u(0,x) + v\,'(x) = v(x),$$

or $$s\, v(x) - \sin(x) + v\,'(x) = v(x).$$

This is an ODE.

ODE: $s\, v(x) - \sin(x) + v\,'(x) = v(x),\ v(0) = 0.$

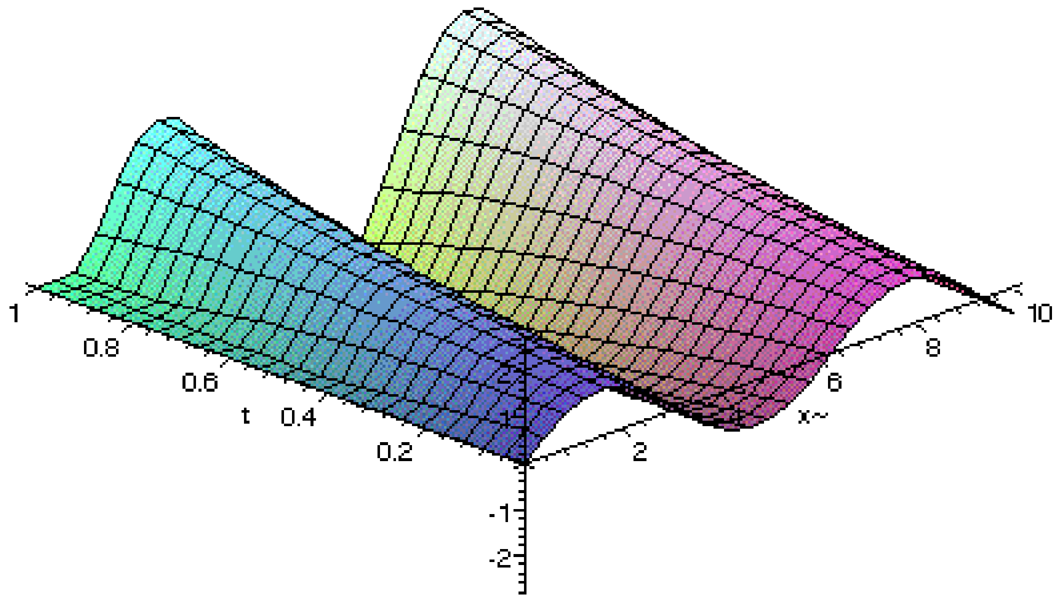
has solution

$(s-1) \sin(x) - \cos(x) + \exp(-(s-1)\, x)\ ] / (s^2 - 2\, s + 2)$

This has inverse laplace transform

$[\ \sin(x - t) - \mathrm{Heaviside}(t - x) \sin(x - t)\ ]\ \exp(\ t)$

Solution for PDE3



In thinking about the character of the graphs of solutions for these three partial differential equations, it is well to compare the character of solutions for

$y' = 0$ goes with $u_t + u_x = 0$,
$y' = -y$ goes with $u_t + u_x = -u$,
$y' = y$ goes with $u_t + u_x = u$.

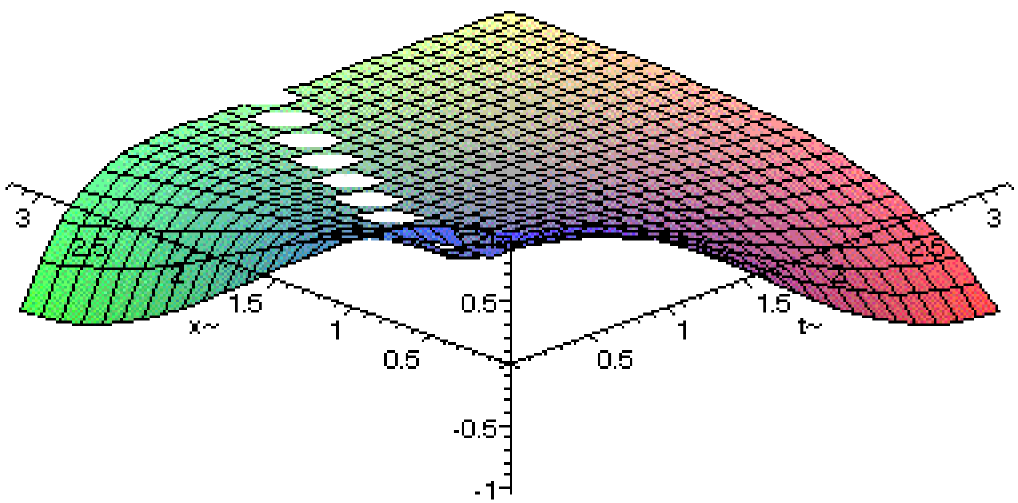
In the accompanying worksheet, there is solved one more PDE. It is different primarily in one manner: for illustration, I also solve the ordinary differential equation by Laplace Transforms.

Here is the final problem:

$$u_t + 2\, u_x = -\,3\, u(t, x),$$

$$u(0, x) = \cos(x),\ u(t, 0) = \cos(t)$$

Solution for PDE4



Assignment: See the Maple Worksheet.

In this Module 40, we have solved first order partial differential equations with constant coefficients using the techniques of Laplace Transforms.