Module 41: Laplace Transforms and the Diffusion Equation

What is possible using Laplace Transforms on the diffusion equation that was not possible previously?

Answer: we have the equation for x in the infinite interval $[0, \infty)$. Previously, we needed two point boundary conditions. We present several examples.

Here is the equation:

$$u_t = k\ u_{xx}$$

with boundary conditions

$u(t,0) = 10$ and $u(t, \infty)$ is bounded.

The initial condition is $u(0,x) = 0$.

The number k is the thermal diffusivity.

We take k to be 1/5.

$$u_t = k\ u_{xx}$$

transforms as

$$s\ v(x) - u(0, x) = 1/5\ \ v''$$

or, using that u(0, x) = 0.

$$s\ v(x) = 1/5\ \ v''$$

Here is the x = 0 condition:

v(0) is the Laplace Transform of u(t, 0),

so that the ODE becomes

$$s\, v(x) = 1/5\ v'', \quad \text{with } v(0) = 10/s\ .$$

I can solve that differential equation in my head:

$$v(x) = 10/s\ \cosh(\sqrt{5s}\ x) + C\ \sinh(\sqrt{5s}\ x).$$

$v(x) = 10/s \cosh(\sqrt{5s}\ x) + C \sinh(\sqrt{5s}\ x)$.

We choose C so that the solutions remain bounded. To accomplish this choosing, change these hyperbolic trig functions back to exponentials.

$(C/2+5/s) \exp(\sqrt{5s}\ x) + (-C/2+5/s) \exp(-\sqrt{5s}\ x)$.

Choose C to be -10/s.

Substitute this C back into v(x):

$$v(x) = 10/s \cosh(\sqrt{5s}\ x) - 10/s \sinh(\sqrt{5s}\ x)$$

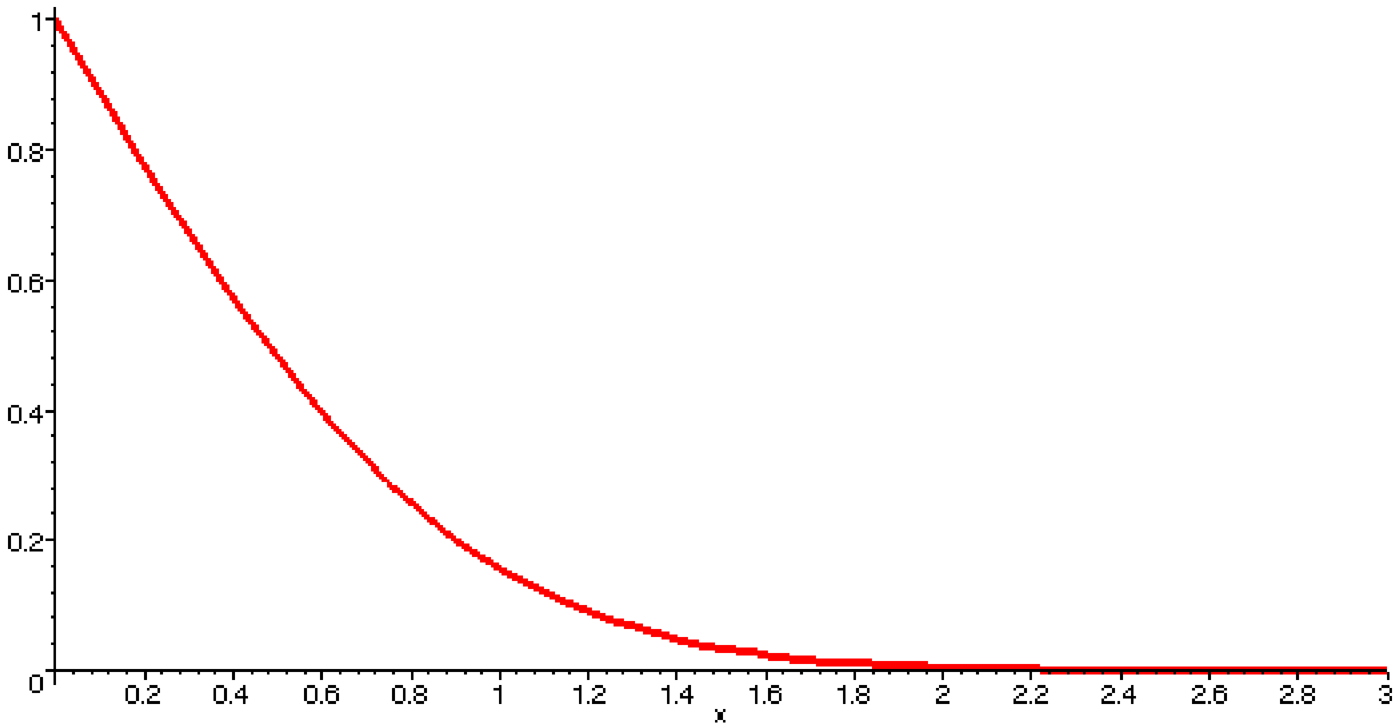
$$= 10/s \exp(-\sqrt{5s}\ x).$$

But, v(x) is the Laplace Transform of u(t, x). Thus, we compute the inverse Laplace Transform. The result is

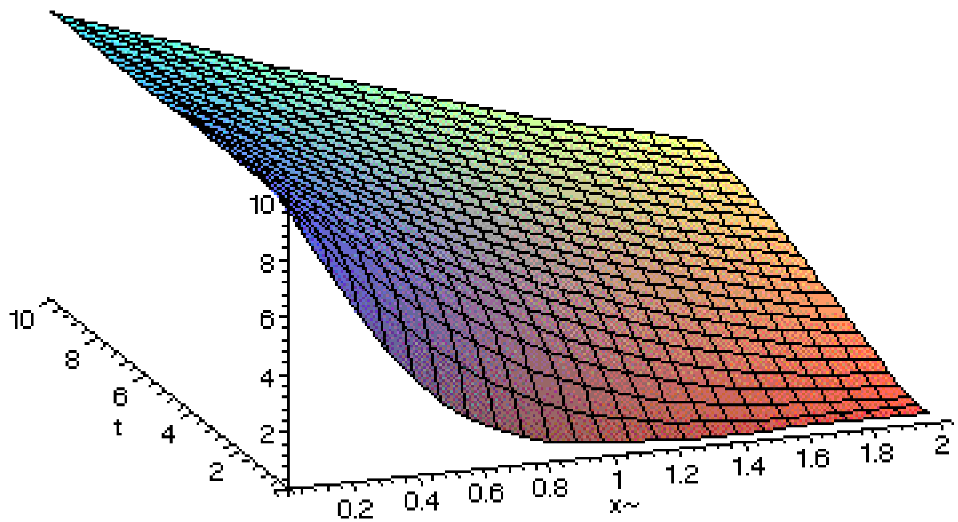
$$u(t, x) = 10 \operatorname{erfc}\left(\frac{x\sqrt{5}}{2\sqrt{t}}\right).$$

This erfc is the complementary error function and is defined by

$$\mathrm{erfc}(x) = 1 - 2/\sqrt{\Pi} \int_0^x \exp(-t^2)\, dt.$$



Now, we can draw the graph of u



Here is the second problem.

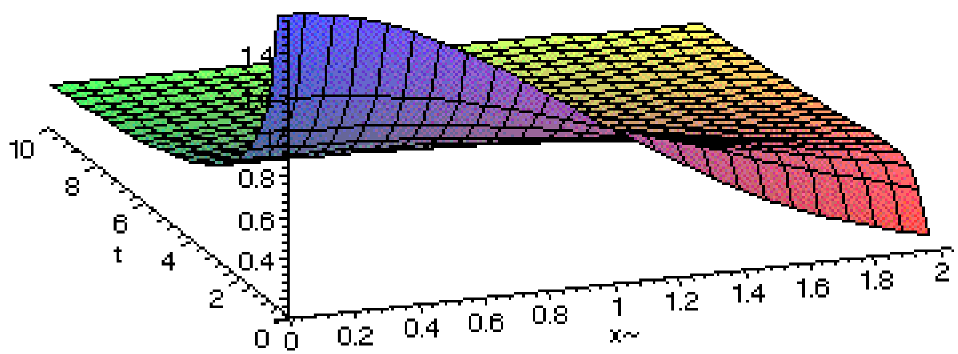
$$u_t = k\, u_{xx}$$

with boundary conditions

$$u(t,0) = 1/\sqrt{t} \quad \text{and } u(t, \infty) \text{ is bounded.}$$

The initial condition is $u(0,x) = 0$.

This problem is not conceptually different. Here is a graph of the solution.



Assignment: See the Maple worksheet.

In this Module 41, we have examined the diffusion equation on an infinite strip in x by using the techniques of Laplace Transforms.