Module 42: Laplace Transforms And The Wave Equation

We do three problems in the accompanying MAPLE worksheet. The first two are similar in description and in solution. We illustrate only the first of these two here. Both these problems can be worked with the methods of d'Alembert. For the third problem, however, it seems more appropriate to use Laplace Transforms. We discuss this one also.

Problem 1: Sending a wave

We have done a problem such as the following but using d'Alembert's techniques. We re-visit this problem with the techniques of Laplace Transforms. Here is the equation:

$$u_{tt} = 9\, u_{xx}, \quad \text{with } u(0, x) = u_t(0, x) = 0,$$
$$\text{and } u(t, 0) = \sin(t),\; u(t, \infty) = 0.$$

The Laplace Transform of the PDE is

$s^2 v(x) - s v(0) - v'(0) = 9 v''(x)$,

where v(x) is the Laplace transform of u(t,x) with respect to t.

This calculation asks for the initial conditions, which will be $v(0) = v'(0) = 0$.

The result: $s^2 v(x) = 9 v''(x)$,

$s^2 v(x) = 9 v''(x)$, with $v(0)$ equal to the Laplace Transform of $\sin(t)$, so that

$$v(0) = 1/(s^2 + 1).$$

How do we make the other condition: $v'(0)$?

It should be made so that solutions remain bounded. Define $v'(0) = b$, and determine how to choose b so that solutions remain bounded.

The solution for the equation, with initial conditions is

something *exp(s x/3) plus

something else* exp(- s x/3).

Choose b so that the first something is zero.

The coefficient b turns out to be $-s/(3(s^2+1))$.

Re-solve the ODE with this value and get

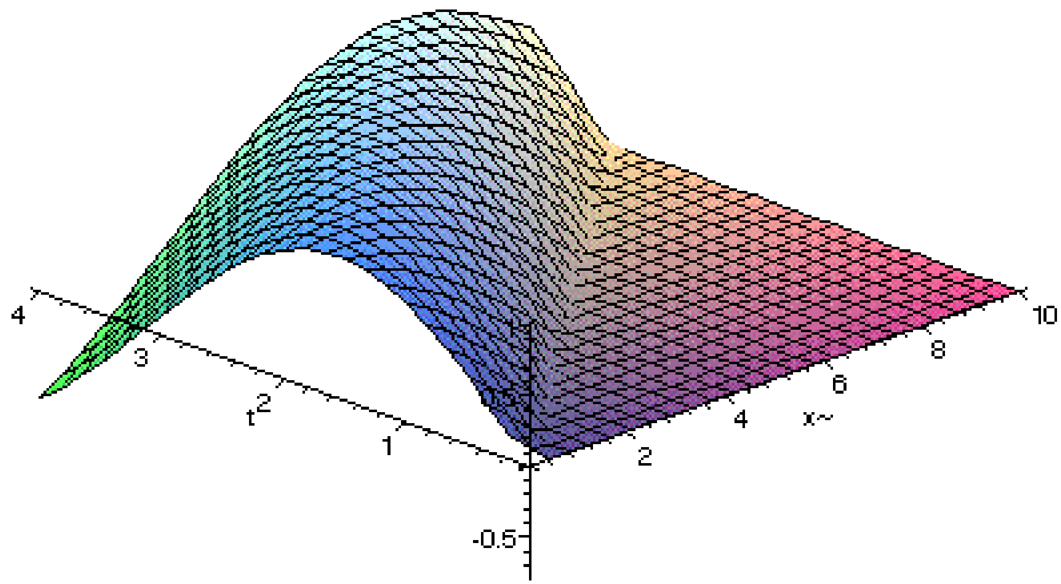
$$v(x) = \exp(- s\ x/3)/ (s^2 + 1).$$

The inverse Laplace Transform of this is

$$u(t, x) = \text{Heaviside}(t - x/3) \sin(t - x/3).$$

We draw the graph.

Graph of Heaviside(t - x/3) sin(t - x/3).



Problem 3: A Long String Falling Under its Weight

This problem is different. It is a model for a string falling due to gravity. The techniques of Laplace Transforms seem more appropriate for this problem than the techniques of separation of variables.

$$u_{tt} - c^2 u_{xx} + g = 0 \qquad \text{with } u(0, x) = u_t(0, x) = 0,$$
$$\text{and } u(t, 0) = 0.$$

The Laplace Transform of the PDE is

$$s^2 v(x) - s\, v(0) - v'(0) - c^2 v''(x) + g/s = 0,$$

where v(x) is the Laplace transform of u(t,x) with respect to t.

This calculation asks for the initial conditions, which will be $v(0) = v'(0) = 0$.

The result: $s^2 v(x) - c^2 v''(x) + g/s = 0$,

$s^2 v(x) - c^2 v''(x) + g/s = 0$, with $v(0)$ equal to the Laplace Transform of 0, so that

$$v(0) = 0.$$

How do we make the other condition: $v'(0)$?

It should be made so that solutions remain bounded. Define $v'(0) = b$, and determine how to choose b so that solutions remain bounded.

The solution for the equation, with initial conditions is

something *exp(s x/c) plus
    something else* exp(- s x/c) - g/s^3.

Choose b so that the first something is zero.

The coefficient b turns out to be - $g/(c\ s^2)$.

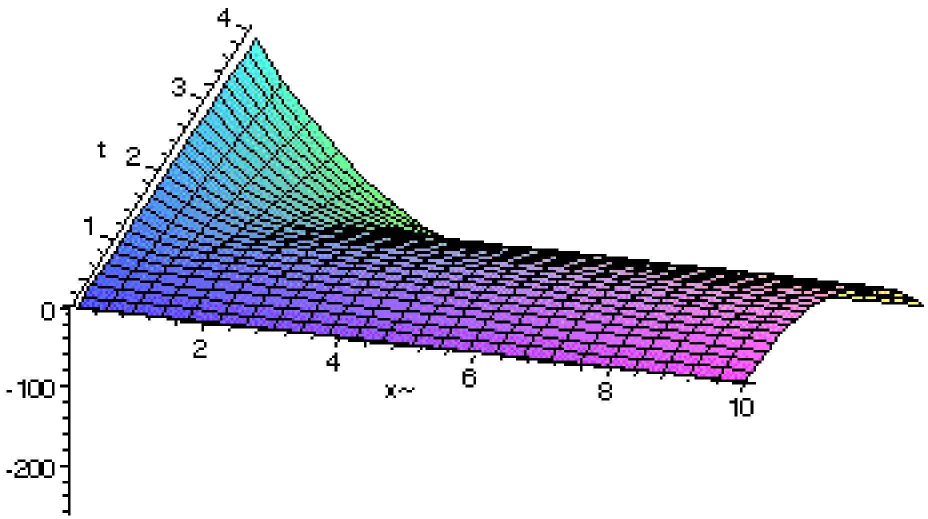
Re-solve the ODE with this value and get

$$v(x) = \exp(s\ x/c)\ g/\ s^3 - g/s^3$$

With g = 32 and c = 1, the inverse Laplace Transform can be computed.

Here is a graph of the resulting solution.

Graph for a dropping string.



Assignment: See the MAPLE worksheet.

In this Module 42, we have used Laplace Transforms to solve the wave equation. For one of the situations Laplace Transforms techniques were most appropriate.