Module 43: Vibrations of a Rigid Beam

David L. Powers in his book, Boundary Value Problems, Third Edition (Published by Harcourt Brace Jovanovich Publishers) gives a problem concerning the vibrations of a rigid beam. The text states that the displacement $u(t, x)$ of a uniform thin beam satisfies

$$u_{xxxx} = -1/c^2 \; u_{tt}, \text{ for } 0 < x < L \text{ and } t > 0.$$

The ends of the beam are simply supported, which produces boundary conditions

$$u(t, 0) = u(t, L) = 0 \quad \text{and } u_{xx}(t, 0) = u_{xx}(t,0) = 0.$$

A derivation of this equation can be found in many texts on undergraduate partial differential equations. See for example Donald W. Trim's Applied Partial Differential Equations published by PWS KENT Publishing Company.

It is natural to ask what is the difference between the transverse vibrations of a string and of a thin beam. An over simplified response would be that the beam offers resistance to bending. This resistance is responsible for changing the wave equation to the fourth order beam equation above.

The constant c incorporates the rigidity and the linear density of the beam.

$$u_{xxxx} = -1/c^2 \ u_{tt}$$

$u(t, 0) = u(t, L) = 0$ and $u_{xx}(t, 0) = u_{xx}(t,0) = 0.$

As for the boundary conditions, simply fastened is usually taken to mean that the ends of the beam are held stationary, but the slopes at the end points can move.

One describes the remaining boundary conditions in terms of the bending moment of the beam. A simply fastened beam should have zero bending moments at the end.

All that remains now is to have the initial conditions:

$$u(0, x) = f(x) \quad \text{and} \quad u_t(0, x) = g(x).$$

We derive the general solution.

We expect separation of variables to lead to solutions of the form

$$u(t, x) = X(x) [A \cos(\lambda t) + B \sin(\lambda t)].$$

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That is, we expect vibrations in the time variable t.

In this problem, separation of variables will lead to equations

$$X''''/X = \lambda^2 = - T''/c^2 T.$$

For the X function:

$$X'''' - \lambda^2 X = 0.$$

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We seek the general solution of this equation. Ask Maple to solve such an equation without any initial conditons and it will give a combination of sines, cosines, hyperbolic sines, and hyperbolic cosines of $\sqrt{\lambda}\ x$. For simplicity, we write

$$X(x) = C\cos(\sqrt{\lambda}\ x) + D\sin(\sqrt{\lambda}\ x) + E\cosh(\sqrt{\lambda}\ x) + F\sinh(\sqrt{\lambda}\ x).$$

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As usual, to determine these constants, we apply the boundary conditions.

$u(t, 0) = 0$ implies that $X(0) = 0$, which implies that $C + E = 0$.

$u_{xx}(t, 0) = 0$ implies that $X''(0) = 0$, which implies that $-C + E = 0$.

$C + E = 0$ and $-C + E = 0$.

From this, we conclude that $C = E = 0$.

We ask what the implications are from the other end boundary conditions.

$u(t, L) = 0$ implies $X(L) = 0$, which implies that

$$D \sin(\sqrt{\lambda}\ L) + F \sinh(\sqrt{\lambda}\ L) = 0.$$

$u_{xx}(t, L) = 0$ implies that $X''(L) = 0$, which implies that

$$-D \sin(\sqrt{\lambda}\ L) + F \sinh(\sqrt{\lambda}\ L) = 0.$$

$$D \sin(\sqrt{\lambda}\ L) + F \sinh(\sqrt{\lambda}\ L) = 0$$

and

$$- D \sin(\sqrt{\lambda}\ L) + F \sinh(\sqrt{\lambda}\ L) = 0.$$

We conclude that $F \sinh(\sqrt{\lambda}\ L) = 0$, so that $F = 0$, and

$$D \sin(\sqrt{\lambda}\ L) = 0, \text{ so that } \sin(\sqrt{\lambda}\ L) = 0.$$

We know everywhere the sine function is zero:

$$\lambda = (n\,\pi / L)^2.$$

This means that there is an infinity of solutions for the X equation and they all have the form

$$X(x) = \sin(\sqrt{\lambda}\ x) = \sin(n\pi / L\ x).$$

The equation in T is easier. First, there are no boundary conditions on the T equation, and second, it is only second order. The equation is

$$(n\pi / L)^4 = - T''/c^2 T.$$

Thus, $$T'' + c^2 (n\pi / L)^4 T = 0.$$

We solve this equation.

$$T'' + c^2 (n\pi / L)^4 T = 0.$$

We can now write down the general solution for the partial differential equation:

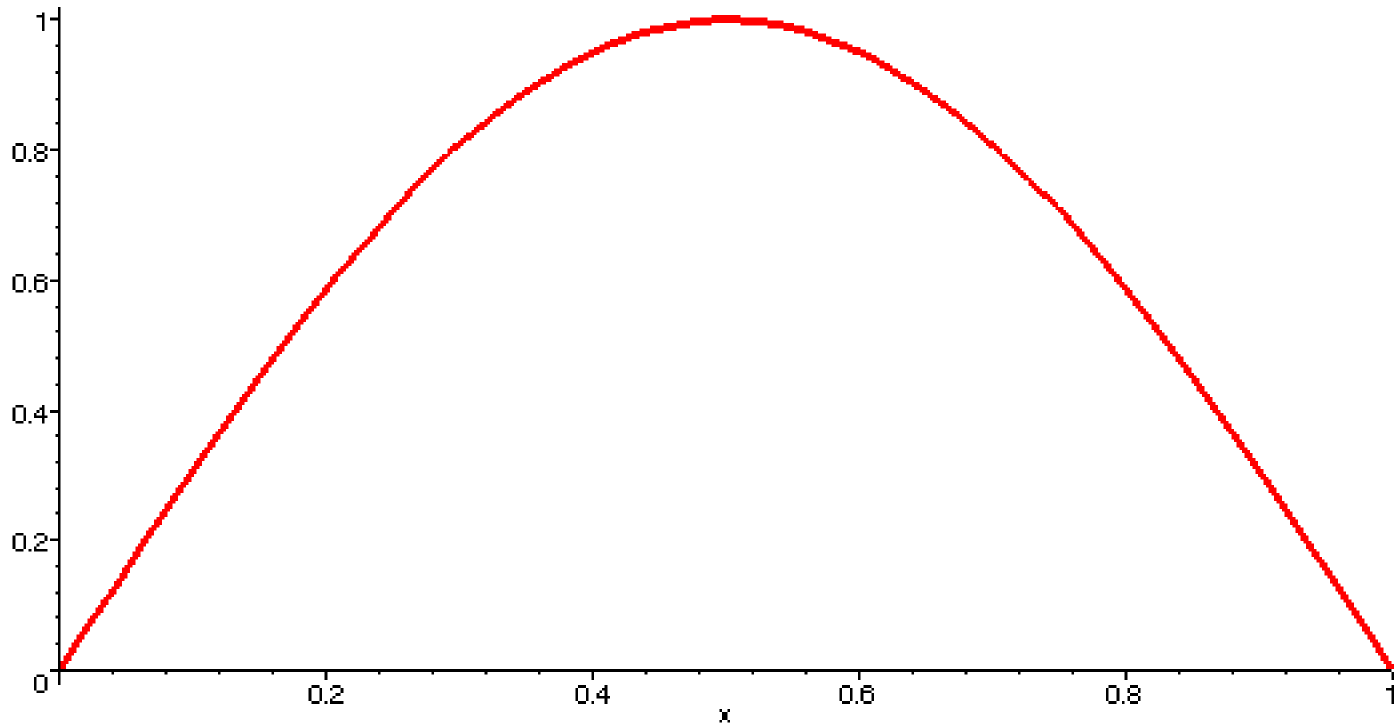
Solutions are cosine and sine of $c (n\pi / L)^2 t$.

Hence, the general solution is

$$u(t, x) = \sum [(a_n \cos(c (n\pi / L)^2 t) + b_n \sin(c (n\pi / L)^2 t)] \sin(n\pi x/L).$$

We can make a check.

Comparison with a string. Initial displacement.



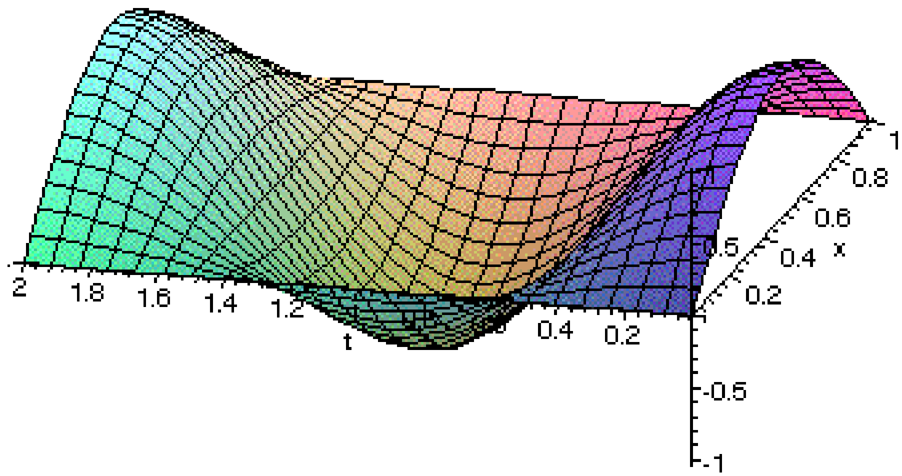
To keep the two the same, we make $c = 1$ so that we are comparing solutions for

$$u_{tt} = u_{xx} \quad \text{and} \quad u_{tt} + u_{xxxx} = 0.$$

Check that the solution for the string equation with zero boundary conditions and no initial velocity is

$$\sin(\pi x) \cos(\pi t).$$

We graph this solution.



$u_{xxxx} = -1/c^2 \; u_{tt}$, for $0 < x < L$ and $t > 0$.

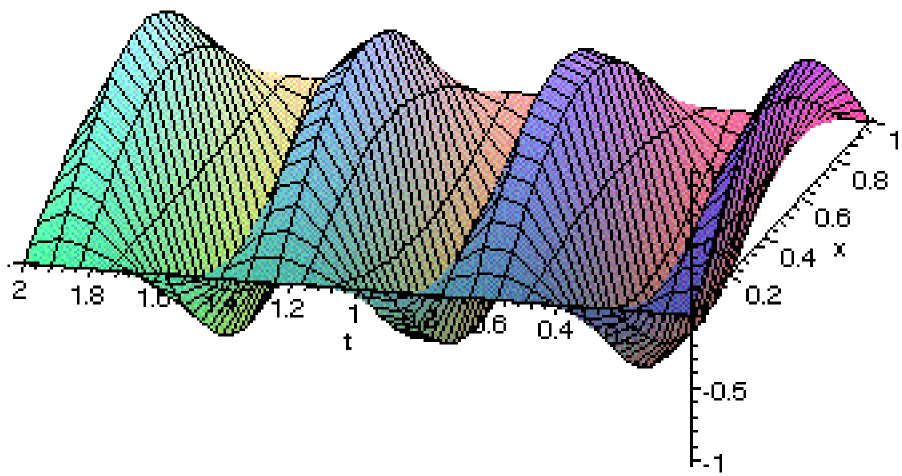
$u(t, 0) = u(t, L) = 0$ and $u_{xx}(t, 0) = u_{xx}(t,0) = 0$.

$u(0, x) = \sin(\pi x)$, $u_t (0, x) = 0$.

Check that the solution for this beam equation with zero boundary conditions and no initial velocity is

$$\sin(\pi x) \cos(\pi^2 t).$$

We graph this solution.



You see the difference with the MAPLE animation. The beam vibrated faster.

$\sin(\pi x)\cos(\pi t)$ and $\sin(\pi x)\cos(\pi^2 t)$.

The string completes one cycle at $t = 2, 4, 6, 8, \ldots$. Watch.
The beam completes one cycle at $t = 2/\pi, 4/\pi, 6/\pi$.

This suggests that if a beam and a string are both struck, parameters for the two being equal, the beam should vibrate at a high pitch.

Assignment. See the Maple worksheet.

In this Module 43, we have compared solutions for a vibrating beam with those for a vibrating string.