Module 44: Forcing Functions

The problem to be solved:

$$\frac{\partial}{\partial t} u = \left(\frac{\partial^2}{\partial x^2} u \right) + \mathrm{F}(t, x)$$

$$\mathrm{u}(t, 0) = 0, \mathrm{u}(t, 1) = 0$$

$$\mathrm{u}(0, x) = \mathrm{h}(x) \ .$$

We suppose that F and h are known.

The problem is to find u.

$$\frac{\partial}{\partial t} u = \left(\frac{\partial^2}{\partial x^2} u \right) + \mathrm{F}(t, x)$$

$$\mathrm{u}(t, 0) = 0, \mathrm{u}(t, 1) = 0$$

A(f) = f '' with f(0) = 0 = f(1)

Step 1: Find eigenvalues and eigen functions for appropriate Sturm Liouville Problems.

Result:

$$\lambda_n = -n^2 \pi^2$$

and $\phi_n(x) = \sin(n \pi x)$.

Step 2: Make a decomposition of h and F as

$$h(x) = \sum c_n \phi_n(x)$$

and $F(t,x) = \sum g_n(t) \phi_n(x)$.

Result:

$$c_n = \frac{\int h(x)\, \phi_n(x)\, dx}{\int \phi_n(x)^2\, dx}$$ and

$$g_n(t) = \frac{\int F(t,x)\, \phi_n(x)\, dx}{\int \phi_n(x)^2\, dx}$$

Step3: Suppose that u(t,x) = $\sum_n T_n(t)\,\phi_n(x)$ and determine an infinite system of equations that define the funtion $T_n(t)$.

Result: $\sum \left(\frac{\partial}{\partial t} T_n(t)\right) \sin(n\,\pi\,x)$ =

$$\sum (-n^2\,\pi^2\,T_n(t))\,\sin(n\,\pi\,x) + \sum g_n(t)\,\sin(n\,\pi\,x).$$

$$\frac{\partial}{\partial t} T_n(t) = -n^2\,\pi^2\,T_n(t) + g_n(t).$$

Step 4: Solve as many of these equations as you have need, time, and money.

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> for n from 1 to N do
    denomin:=int(sin(n*Pi*x)^2,x=0..1):
    T0:=int(u0(x)*sin(n*Pi*x),x=0..1)/denomin:
    gn:=t->int(F(t,x)*sin(n*Pi*x),x=0..1)/denomin:
    eq:=diff(Y(t),t)=-n^2*Pi^2*Y(t)+gn(t):
    sol:=dsolve({eq,Y(0)=T0},Y(t)):
    T[n]:=unapply(rhs(sol),t);
  od:
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Step 5: Formulate the solution and check your results:

u(t,x) = $\sum_n T_n(t) \sin(n \pi x)$.

We think about how to check the results

1. Check with the PDE,
2. Check with the boundary conditions,
3. Check with the initial conditions by drawing a graph.

$$\frac{\partial}{\partial t} u = \left(\frac{\partial^2}{\partial x^2} u \right) + F(t, x)$$

$$\left(\frac{\partial}{\partial x} u \right)(t, 0) = 0, \left(\frac{\partial}{\partial x} u \right)(t, 1) = 0$$

Now, the Sturm Liouville problem is

A(f) = f '' with f '(0) = 0 = f '(1).

This has eigenvalues

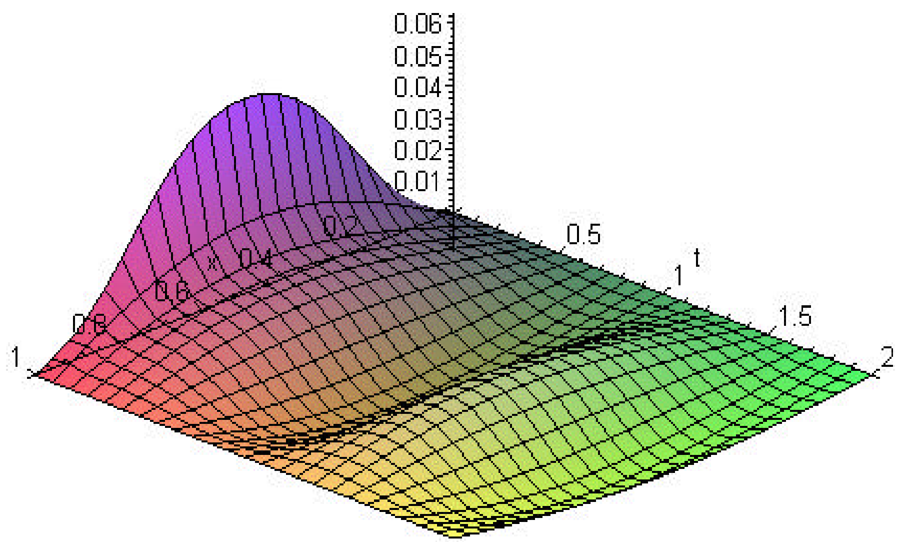
$$\lambda_n = -n^2 \pi^2 \text{ , n= 0, 1, 2, ...}$$

and $\phi_n(x) = \cos(n \pi x)$.

Here is an example:

$h(x)=x^2\,(1-x)^2$

$F(t,x) = \sin(2\pi t)\, x^2\,(1-x)$

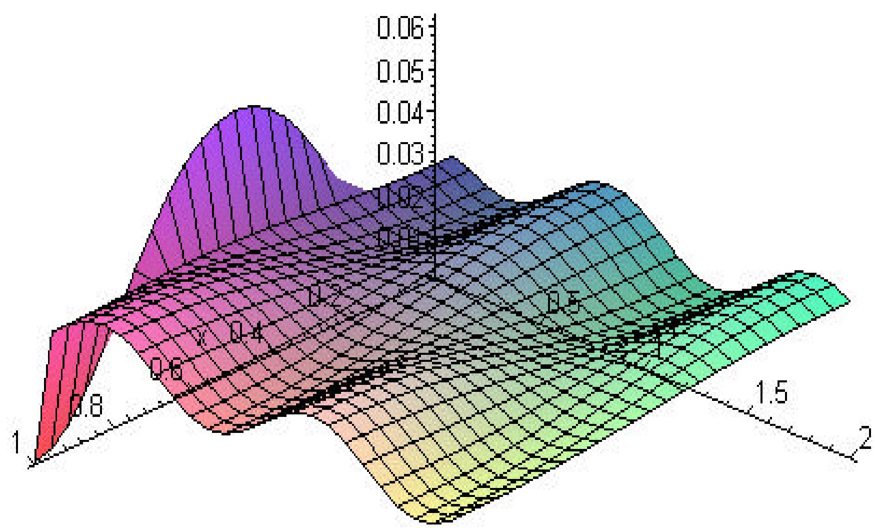


Observe a not so subtle feature of this graph.

$h(x) = x^2 (1-x)^2$

$F(t,x) = \sin(2 \pi t)\, x\, (1-x)\, (2x-1)$

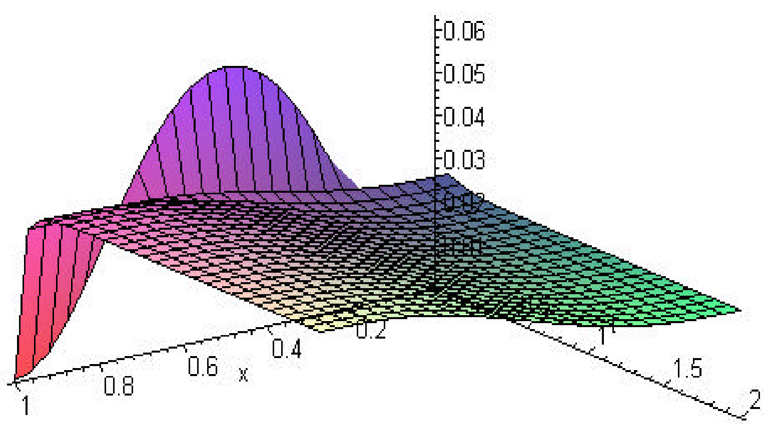
Features ?



$h(x) = x^2 \, (1-x)^2$

$F(t,x) = x \, (1-x) \, (2x-1)$

Features ?



Summary: We solved first order differential equations with a non-constant forcing function in three contexts:

a. the calculus,

b. ordinary differential equations, and

c. partial differential equations.