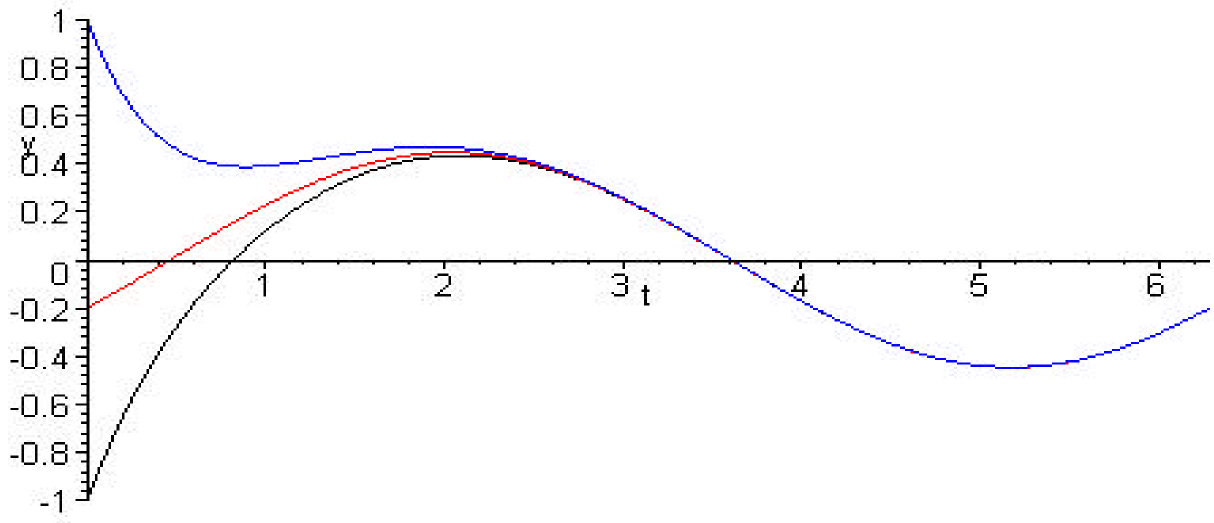
Module 45: Periodic Forcing Functions

Recall the problem

$$Y'(t) = -2\,Y(t) + \sin(t).$$

There is a periodic forcing function.

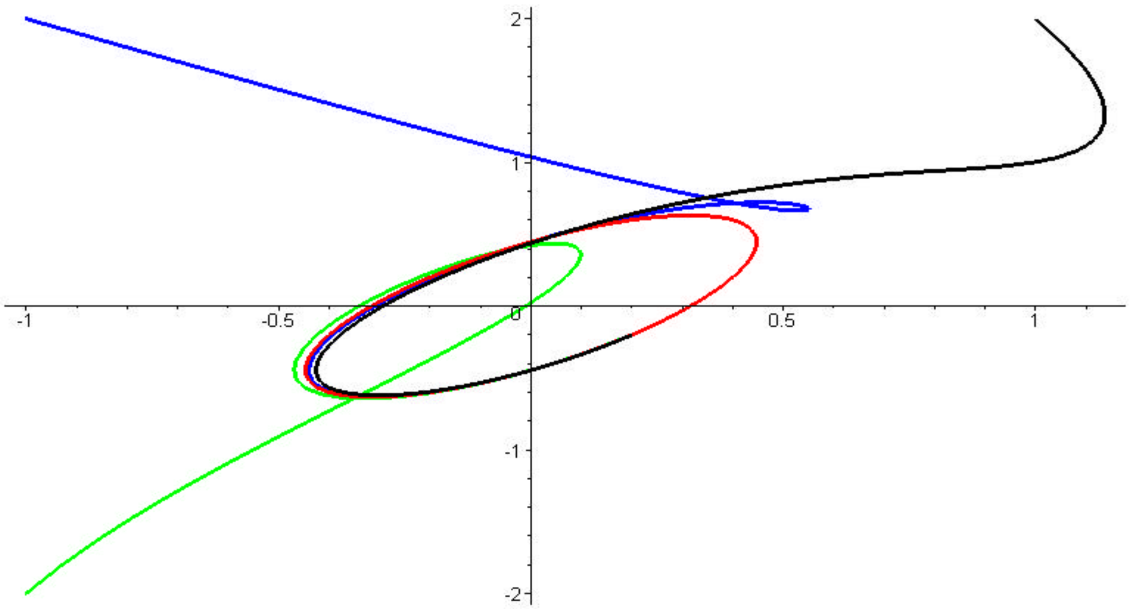


Recall the problem

$x'(t) = 2\,x(t) + y(t) + \cos(t)$

$y'(t) = x(t) - 2\,y(t) + \sin(t)$

There is a periodic forcing function.

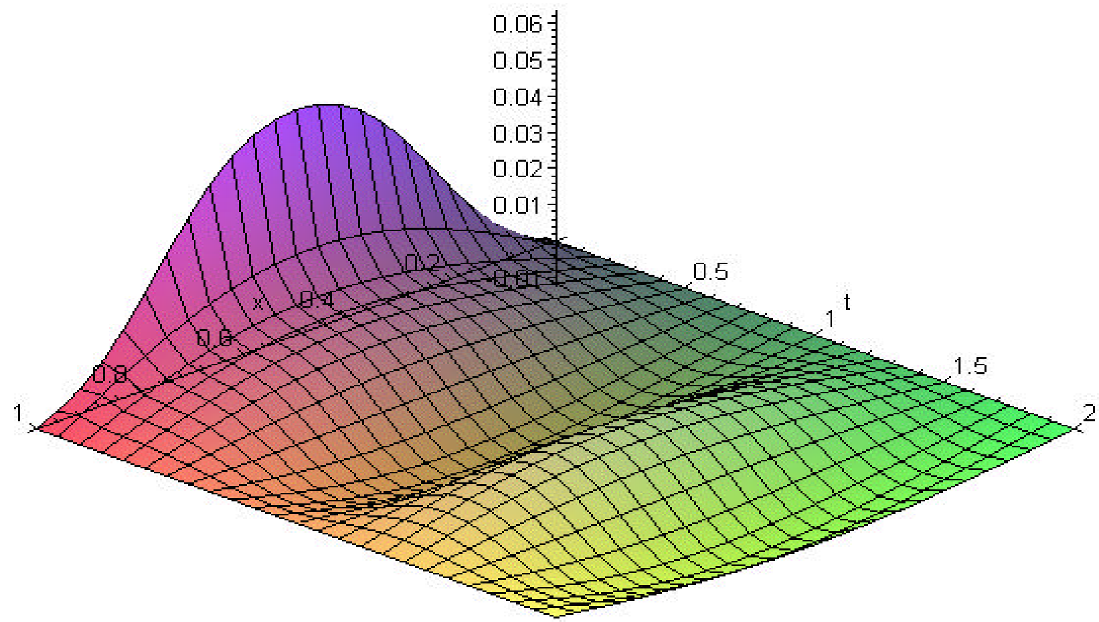


Recall the problem

$$du/dt = d^2u/dx^2 + \sin(2\pi t)\, x^2 (1-x)$$

$$u(t,0) = 0 = u(t,1)$$

There is a periodic forcing function.



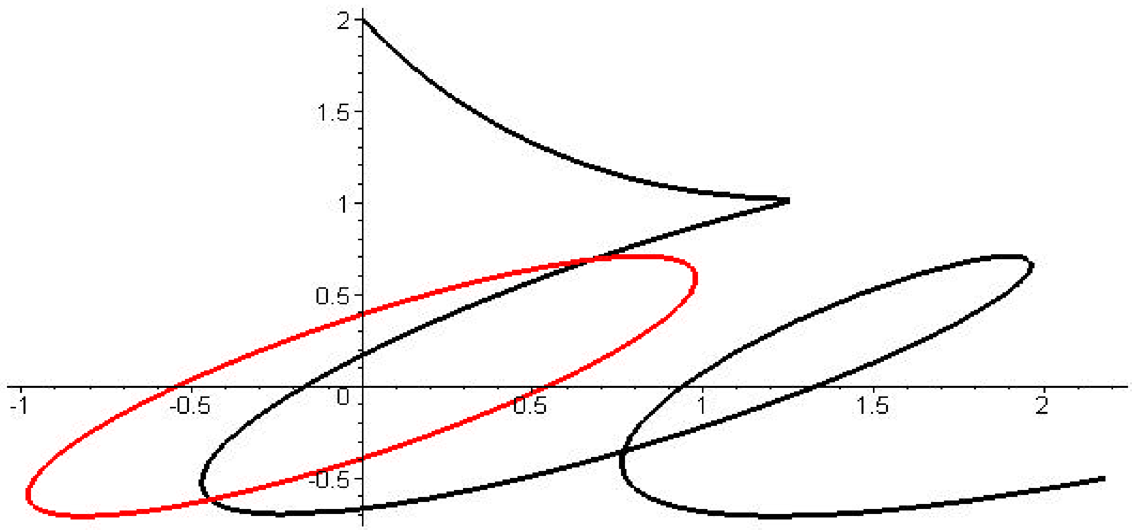
What these three examples have in common is that there is a periodic forcing function. It appears that two other things happen: one is that there is a periodic solution for the equation and the other is that all other solutions converge to this periodic solution.

We would like to be able to find the periodic solution.

What is needed is the initial value for the solution that will be periodic.

Before showing how to get the initial value for a periodic solution, consider this example:

$$x'(t) = 0.2\,x(t) + \cos(t), \quad y'(t) = -\,y(t) + \sin(t).$$



Theorem: Suppose that c is a number not zero and that f is a piecewise continuous, periodic function with period P. There is a number z_0 such that the solution for

$$Z'(t) = c\, Z(t) + f(t), \text{ with } Z(0) = z_0 \qquad (*)$$

is periodic. Moreover, if Y is any other solution of (*), then

$$|Y(t) - Z(t)| <= e^{c\, t}\, |Y(0) - Z(0|$$

so that solutions have limit the periodic solution or move away from it depending as to whether c < 0 or c > 0.

Review Lecture 39 for how to get periodic solutions: if f(t) is periodic with period P then solve z' = a z + f, with f(0) = c. Determine the initial value c by requiring that z(P) = z(0).

This idea is illustrated for the first two examples above in the accompanying Maple worksheet.

The partial differential equation setting is more interesting.

Consider the diffusion equation with zero boundary conditions and with a periodic forcing function:

$$du/dt = d^2u/dx^2 + \sin(2\pi\, t)\ x^2\ (1-x)$$

$$u(t,0) = 0 = u(t,1).$$

The associated Sturm-Liouville operator is

$$A(y) = y'' \text{ with } y(0) = 0 = y(1).$$

What are the eigenvalues and eigenfunctions?

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Answer: eigenvalues are $- n^2 \pi^2$
and eigenfunctions are $\sin(n \pi x)$.

Expand $f(t,x)$ as $\Sigma_n\, g_n(t) \sin(n \pi x)$, and

$$U(t,x) = \Sigma_n\, T_n(t) \sin(n \pi x).$$

Substitute these into the equation:

$$du/dt = d^2u/dx^2 + \sin(2\pi t)\, x^2\, (1-x)$$

$$\mathbf{S}_n \; T_n'(t) \sin(n \pi x) = \mathbf{S}_n \; [- n^2 \pi^2 T_n(t) + g_n(t)] \sin(n \pi x)$$

There is an infinite set of ordinary differential equations:

For each n,

$$T_n'(t) = - n^2 \pi^2 T_n(t) + g_n(t).$$

What is the initial value for these? Remember how to get periodic solutions for ordinary differential equations. Choose c so that

$$T_n(P) = T_n(0).$$

For each n, DO the following:

1. Compute $g_n(t)$.
2. Write down the ODE.
3. Solve the ODE with initial value c.
4. Choose c so that the solution is periodic.
5. Call the resulting solution $T_n(t)$.

Make up

$$U(t,x) = \Sigma_n \; T_n(t) \sin(n \pi x).$$

Check the solution:

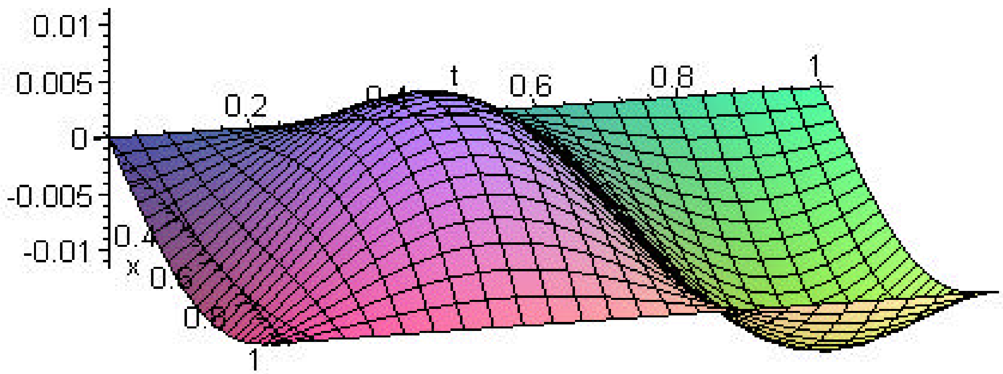
Check boundary conditions,

Check the partial differential equation,

Check graphically that the solution is periodic.

Draw the graph of u. In case

$$F(t, x) = \sin(2 \pi t)\, x^2 (1-x),$$



Summary: We have found periodic solutions for differential equations which have periodic forcing functions in three contexts:

a. the calculus,
b. the ordinary differential equations, and
c. the partial differential equations.